CONTINUOUS MAPPINGS WITH NULL SUPPORT

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Abstract. Let X be a topological space and let $\mathscr{I}$ be an ideal in X. (That is, $\mathscr{I}$ is a collection of subsets of X such that every subset of an element of $\mathscr{I}$ is in $\mathscr{I}$ and the union of any two elements of $\mathscr{I}$ is in $\mathscr{I}$.) The elements of $\mathscr{I}$ are called *null*. The space X is *locally null* if each of its points has a null neighborhood in X.

We introduce and study the normed subalgebra $C_{00}^{\mathscr{I}}(X)$ of $C_b(X)$ consisting of those $f \in C_b(X)$ whose support has a null neighborhood in X, and the Banach subalgebra $C_0^{\mathscr{I}}(X)$ of $C_b(X)$ consisting of those $f \in C_b(X)$ such that $|f|^{-1}([1/n,\infty))$ has a null neighborhood in X for each positive integer n. In particular, we prove that if X is a normal locally null space then $C_{00}^{\mathscr{I}}(X)$ and $C_0^{\mathscr{I}}(X)$ are respectively isometrically isomorphic to $C_{00}(Y)$ and $C_0(Y)$ for some unique locally compact Hausdorff space Y. Furthermore, $C_{00}^{\mathscr{I}}(X)$ is dense in $C_0^{\mathscr{I}}(X)$. We construct Y as a subspace of the Stone–Čech compactification βX of X. The space Y is locally compact, contains X densely, and is sometimes countably compact. We identify Y as familiar subspaces of βX in specific cases. The known construction of Y enables us to better study $C_{00}^{\mathscr{I}}(X)$ and $C_0^{\mathscr{I}}(X)$ and derive some of their properties. This is particularly done when we consider specific examples of either the space X or the ideal $\mathscr{I}$.

Contents

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1. Introduction

Throughout this article by a *space* we mean a topological space. Let X be a space. We denote by $C(X)$ the set of all continuous $f : X \to \mathbb{R}$ and we denote by $C_b(X)$ the set of all bounded elements of $C(X)$. If $f \in C(X)$ the *support* of f is

$$\operatorname{supp}(f) = \operatorname{cl}_X\big\{x \in X : f(x) \neq 0\big\}.$$

We denote by $C_0(X)$ the set of all $f \in C_b(X)$ which vanish at infinity (that is, $|f|^{-1}([1/n, \infty))$ is compact for each positive integer n) and we denote by $C_{00}(X)$ the set of all $f \in C_b(X)$ with compact support.

An *upper semi-lattice* $(L, \leq)$ is a partially ordered set that together with any two elements $a, b \in L$ it contains their least upper bound $a \vee b$. Let $(L, \leq)$ be an upper semi-lattice. A non-empty subset I of L is an *ideal* in L if it satisfies the following:

- If $a, b \in I$ then $a \vee b \in I$.
- If $t \in L$ and $t \leq a \in I$ then $t \in I$.

Suppose that $(L, \leq)$ contains the least upper bound for any countable number of elements in L. An ideal I of L is a *σ-ideal* in L if it satisfies the following:

- If $a_1, a_2, \ldots \in I$ then $a_1 \vee a_2 \vee \cdots \in I$.

We are particularly interested in upper semi-lattices $(\mathscr{L}, \leq)$ such that $\mathscr{L} \subseteq \mathscr{P}(X)$, where X is a space and $\leq$ is the set-theoretic inclusion $\subseteq$; the elements of $\mathscr{I}$ are then called *null* (or *negligible*). The ideal $\mathscr{I}$ is *proper* if X in not null. The space X is *locally null* if each of its points has a null neighborhood in X. An ideal may be interpreted as a collection of sets that are considered to be somehow "small" or "negligible". Every element of the upper semi-lattice contained in an element of the ideal must also be in the ideal; this codifies the notion of "smallness".

Let X be a locally separable metrizable space. In [16] (and [19]) we have introduced and studied the Banach algebra $C_s(X)$ of all $f \in C_b(X)$ with separable support, using techniques we have already developed in [14] and [18]. Here we aim to follow the same route. Our results here generalize and unify our earlier results in [16], [19] and [20].

The article is divided into two parts.

In the first part, for a space X and an ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L}, \subseteq)$, where $\mathscr{L} \subseteq \mathscr{P}(X)$, we consider the subset $C^{\mathscr{I}}_{00}(X)$ of $C_b(X)$ consisting of those $f \in C_b(X)$ whose support has a null neighborhood in X and the subset $C^{\mathscr{I}}_0(X)$ of $C_b(X)$ consisting of those $f \in C_b(X)$ such that $|f|^{-1}([1/n, \infty))$ has a null neighborhood in X for each positive integer n. Under certain conditions the expression of either $C^{\mathscr{I}}_{00}(X)$ or $C^{\mathscr{I}}_0(X)$ simplifies. In particular

$$C^{\mathscr{I}}_{00}(X) = \big\{f \in C_b(X) : \operatorname{supp}(f) \text{ is null}\big\}$$

if X is Lindelöf and locally null, $\mathscr{I}$ is a σ-ideal and $\mathscr{L}$ contains the set $\mathscr{RC}(X)$ of all regular closed subspaces of X, and

$$C^{\mathscr{I}}_0(X) = \big\{f \in C_b(X) : |f|^{-1}\big([1/n, \infty)\big) \text{ is null for each } n\big\}$$

if $\mathscr{L}$ contains the set $\mathscr{Z}(X)$ of all zero-sets of X. The sets $C^{\mathscr{I}}_{00}(X)$ and $C^{\mathscr{I}}_0(X)$ coincide with $C_{00}(X)$ and $C_0(X)$, respectively, if X is locally compact and $\mathscr{I}$ is the set of all subspaces of X with compact closure. We show that in general $C^{\mathscr{I}}_{00}(X)$ is an ideal in $C_b(X)$ and $C^{\mathscr{I}}_0(X)$ is a closed ideal in $C_b(X)$ containing $C^{\mathscr{I}}_{00}(X)$. Furthermore, if X is completely regular, then $C^{\mathscr{I}}_{00}(X)$ is of empty hull if and only if $C^{\mathscr{I}}_0(X)$ is of empty hull if and only if X is locally null, and if so, then $C^{\mathscr{I}}_{00}(X)$ is

unital if and only if $C_0^{\mathscr{I}}(X)$ is unital if and only if $\mathscr{I}$ is non-proper. The main result of this part states that if X is normal and locally null then $C_{00}^{\mathscr{I}}(X)$ and $C_0^{\mathscr{I}}(X)$ are respectively isometrically isomorphic to $C_{00}(Y)$ and $C_0(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space $Y = \lambda_{\mathscr{I}} X$, where

$$\lambda_{\mathscr{I}} X = \bigcup \big\{\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} C : C \in \mathrm{Coz}(X) \text{ and } \mathrm{cl}_X C \text{ has a null neighborhood in } X\big\}$$

is a subspace of the Stone–Čech compactification βX of X. (Here $\mathrm{Coz}(X)$ denotes the set of all cozero-sets of X.) Furthermore, $C_0^{\mathscr{I}}(X)$ contains $C_{00}^{\mathscr{I}}(X)$ densely, Y contains X densely, and Y is compact if and only if $C_{00}^{\mathscr{I}}(X)$ is unital if and only if $C_0^{\mathscr{I}}(X)$ is unital. Moreover, X is also Lindelöf and $\mathscr{I}$ is a σ-ideal, then Y is countably compact and $C_0^{\mathscr{I}}(X) = C_{00}^{\mathscr{I}}(X)$.

In the second part, we consider specific examples. This specification, either of the space X or the ideal $\mathscr{I}$, enables us to study $C_{00}^{\mathscr{I}}(X)$ and $C_0^{\mathscr{I}}(X)$ further and deeper. This part is divided into four sections.

In the first section, for a topological measure space $(X, \mathscr{O}, \mathscr{B}, \mu)$ we consider the ideal

$$\mathscr{M} = \big\{B \in \mathscr{B} : \mu(B) = 0\big\}.$$

This leads to the consideration of the Banach subalgebra of $C_b(X)$ consisting of elements with μ-null cozero-set.

In the second section, for a regular Lindelöf space X we consider the closed z-ideals H in $C_b(X)$ with empty hull. We show that such an H contains a Banach subalgebra K of the form $C_0(Y) = C_{00}(Y)$ for some locally compact countably compact Hausdorff space Y. Furthermore, K is unital if and only if H contains an element not vanishing on X.

In the third section we consider certain known ideals of $\mathbb{N}$. These include the summable ideal

$$\mathscr{S} = \bigg\{A \subseteq \mathbb{N} : \sum_{n\in A} \frac{1}{n} \text{ converges}\bigg\}$$

and the density ideal

$$\mathscr{D} = \bigg\{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{\big|A \cap \{1, \ldots, n\}\big|}{n} = 0\bigg\}.$$

of $\mathbb{N}$. If we let

$$\mathfrak{s}_{00} = C_{00}^{\mathscr{S}}(\mathbb{N}), \quad \mathfrak{d}_{00} = C_{00}^{\mathscr{D}}(\mathbb{N}), \quad \mathfrak{s}_0 = C_0^{\mathscr{S}}(\mathbb{N}) \quad \text{and} \quad \mathfrak{d}_0 = C_0^{\mathscr{D}}(\mathbb{N})$$

then

$$\mathfrak{s}_{00} = \bigg\{\mathbf{x} \in \ell_\infty : \sum_{\mathbf{x}(n)\neq 0} \frac{1}{n} \text{ converges}\bigg\},$$

and

$$\mathfrak{d}_{00} = \bigg\{\mathbf{x} \in \ell_\infty : \limsup_{n\to\infty} \frac{\big|\{k \leq n : \mathbf{x}(k) \neq 0\}\big|}{n} = 0\bigg\},$$

are normed subalgebras of ℓ_∞ and

$$\mathfrak{s}_0 = \bigg\{\mathbf{x} \in \ell_\infty : \sum_{|\mathbf{x}(n)|\geq\epsilon} \frac{1}{n} \text{ converges for each } \epsilon > 0\bigg\},$$

and

$$\mathfrak{d}_0 = \bigg\{\mathbf{x} \in \ell_\infty : \limsup_{n\to\infty} \frac{\big|\{k \leq n : |\mathbf{x}(k)| \geq \epsilon\}\big|}{n} = 0 \text{ for each } \epsilon > 0\bigg\}.$$

are Banach subalgebras of ℓ_∞. Moreover, $\mathfrak{s}_{00}$ and $\mathfrak{d}_{00}$ each contains an isometric copy of the normed algebra $\bigoplus_{n=1}^\infty \ell_\infty$, $\mathfrak{s}_{00}/c_{00}$ and $\mathfrak{d}_{00}/c_{00}$ each contains a copy of the algebra

$$\bigoplus_{i<2^\omega}\frac{\ell_\infty}{c_{00}}$$

and $\mathfrak{s}_0/c_0$ and $\mathfrak{d}_0/c_0$ each contains an isometric copy of the normed algebra

$$\bigoplus_{i<2^\omega}\frac{\ell_\infty}{c_0}.$$

Finally, in the fourth section we consider for a space X and a topological property $\mathfrak{P}$ the ideal

$$\mathscr{I}_\mathfrak{P}=\{A\subseteq X:\mathrm{cl}_X A \text{ has } \mathfrak{P}\}$$

of $(\mathscr{P}(X),\subseteq)$. Here $\mathfrak{P}$ is required to satisfy certain mild requirements. We consider $C_{00}^\mathscr{I}(X)$ and $C_0^\mathscr{I}(X)$ where $\mathscr{I}=\mathscr{I}_\mathfrak{P}$; for simplicity of the notation denote them by $C_{00}^\mathfrak{P}(X)$ and $C_0^\mathfrak{P}(X)$, respectively. Also, denote $\lambda_\mathscr{I} X$ by $\lambda_\mathfrak{P} X$. In this context

$$C_{00}^\mathfrak{P}(X)=\big\{f\in C_b(X):\mathrm{supp}(f) \text{ has a closed neighborhood in } X \text{ with } \mathfrak{P}\big\}$$

and

$$C_0^\mathfrak{P}(X)=\big\{f\in C_b(X):|f|^{-1}\big([1/n,\infty)\big) \text{ has } \mathfrak{P} \text{ for each } n\big\}.$$

The ideal $\mathscr{I}_\mathfrak{P}$ is non-proper if and only if X is non-$\mathfrak{P}$, and if X is regular, X is locally null if and only if X is locally-$\mathfrak{P}$. Particular attention will be paid to those spaces X and topological properties $\mathfrak{P}$ such that X is locally-$\mathfrak{P}$ and has $\mathfrak{Q}$ and $\mathfrak{P}$ satisfies

$$\mathfrak{P}+\mathfrak{Q}\rightarrow \text{The Lindelöf property},$$

where $\mathfrak{Q}$ is a topological property subject to some requirements. In particular, in this case we would have

$$C_{00}^\mathfrak{P}(X)=\big\{f\in C_b(X):\mathrm{supp}(f) \text{ has } \mathfrak{P}\big\}.$$

The special case in which $\mathfrak{P}$ is the Lindelöf property and $\mathfrak{Q}$ is metrizability (or paracompactness) is studied in great detail. Among other things, we show that $\lambda_\mathfrak{P} X$ is countably compact and is non-normal if X is non-$\mathfrak{P}$. In particular,

$$\dim C_{00}^\mathfrak{P}(X)=\ell(X)^{\aleph_0},$$

where $\ell(X)$ is the Lindelöf number of X. The concluding results in this section deal with realcompactness and pseudocompactness. We show that if $\mathfrak{P}$ is realcompactness and X is normal, then

$$\lambda_\mathfrak{P} X=\beta X\backslash\mathrm{cl}_{\beta X}(\upsilon X\backslash X)$$

where υX is the Hewitt realcompactification of X. Also, if $\mathfrak{P}$ is pseudocompactness and X is completely regular, then

$$\lambda_\mathscr{U} X=\mathrm{int}_{\beta X}\upsilon X$$

for the ideal

$$\mathscr{U}=\big\{A\in\mathscr{RC}(X):A \text{ is pseudocompact}\big\}$$

of the partially ordered set $(\mathscr{RC}(X),\subseteq)$ of all regular closed subspaces of X.

In the recent preprint [26], for a completely regular space X and a filter base $\mathscr{B}$ of open subspaces of X, the author defined $C_\mathscr{B}(X)$ to be the set of all $f\in C(X)$ whose support is contained in $X\backslash A$ for some $A\in\mathscr{B}$, and $C_{\infty\mathscr{B}}(X)$ to be the set

of all $f \in C(X)$ such that $|f|^{-1}([1/n,\infty))$ is contained in $X\backslash A$ for some $A \in \mathscr{B}$ for each positive integer n. (See [2] for certain special cases.) Also, if $\mathscr{I}$ is an ideal of closed subspaces of X, in [1], the authors defined $C_{\mathscr{I}}(X)$ to be the set of all $f \in C(X)$ whose support is contained in $\mathscr{I}$, and $C^{\mathscr{I}}_{\infty}(X)$ to be the set of all $f \in C(X)$ such that $|f|^{-1}([1/n,\infty))$ is contained in $\mathscr{I}$ for each positive integer n. Despite similarities between our definitions and the definitions given in [26] or [1], the existing differences between definitions have caused this work to move in a different direction, leaving little in common with either [26] or [1].

2. Preliminaries

This section contains certain notion and known facts that we will use frequently throughout this article. The first part is to provide examples of upper semi-lattices; upper semi-lattices are the natural setting to state and prove our results. The second part review certain properties of the Stone–Čech compactification; the Stone–Čech compactification is the main tool in our study. For more information on the theory of the Stone–Čech compactification we refer the reader to [7], [9], [24] and [29].

Examples of upper semi-lattices. In the following we give examples of upper semi-lattices. (Indeed, the examples are all lattices, however, this will not be used in the sequel.)

(1) Let X be a space. A subspace A of X is *regular closed* in X if $A = \mathrm{cl}_X\mathrm{int}_X A$. Denote by $\mathscr{RC}(X)$ the set of all regular closed subspaces of X. Note that the closure of each open subspace of X is regular closed in X. The partially ordered set $(\mathscr{RC}(X),\subseteq)$ is an upper semi-lattice. If $\mathscr{B} \subseteq \mathscr{RC}(X)$ is non-empty then
$$\bigvee \mathscr{B} = \mathrm{cl}_X\Big(\mathrm{int}_X\Big(\bigcup \mathscr{B}\Big)\Big).$$
In particular, if $A, B \in \mathscr{RC}(X)$ then
$$A \vee B = A \cup B.$$

(2) Let X be a space. A subspace A of X is *regular open* in X if $A = \mathrm{int}_X\mathrm{cl}_X A$. Denote by $\mathscr{RO}(X)$ the set of all regular open subspaces of X. Note that the interior of each closed subspace of X is regular open in X. The partially ordered set $(\mathscr{RO}(X),\subseteq)$ is an upper semi-lattice. If $\mathscr{B} \subseteq \mathscr{RO}(X)$ is non-empty then
$$\bigvee \mathscr{B} = \mathrm{int}_X\Big(\mathrm{cl}_X\Big(\bigcup \mathscr{B}\Big)\Big).$$
In particular, if $A, B \in \mathscr{RO}(X)$ then
$$A \vee B = \mathrm{int}_X\mathrm{cl}_X(A \cup B).$$

(3) Let X be a space. A subspace Z of X is a *zero-set* in X if $Z = f^{-1}(0)$ for some continuous $f : X \to [0,1]$. Denote by $\mathscr{Z}(X)$ the set of all zero-sets of X. The partially ordered set $(\mathscr{Z}(X),\subseteq)$ is an upper semi-lattice. If $Z, S \in \mathscr{Z}(X)$ then
$$S \vee Z = S \cup Z;$$
to see this, let $f, g : X \to [0,1]$ be continuous and observe that
$$\mathrm{Z}(f) \cup \mathrm{Z}(g) = \mathrm{Z}(fg).$$

(4) Let X be a space. A subspace C of X is a *cozero-set* in X if $C = X\backslash Z$ for some zero-set Z in X. Denote by $\mathrm{Coz}(X)$ the set of all cozero-sets of X. The partially ordered set $(\mathrm{Coz}(X), \subseteq)$ is an upper semi-lattice. Indeed, if $C, D \in \mathrm{Coz}(X)$ then
$$C \vee D = C \cup D.$$
More generally, if $\mathscr{B} \subseteq \mathrm{Coz}(X)$ is non-empty and countable then
$$\bigvee \mathscr{B} = \bigcup \mathscr{B};$$
as, if $f_n : X \to [0,1]$ is continuous for each positive integer n, then
$$\bigcup_{n=1}^{\infty} \mathrm{Coz}(f_n) = \mathrm{Coz}(f)$$
where
$$f = \sum_{n=1}^{\infty} \frac{f_n}{2^n}.$$
(That f is continuous follows from the Weierstrass M-test.)

The fact that $(\mathscr{L}, \subseteq)$ is an upper semi-lattice does *not* necessarily mean that if $A, B \in \mathscr{L}$ then $A \vee B = A \cup B$. (For instance, let $\mathscr{L} = \mathscr{RO}(X)$ where X is a space. Then $A \vee B = \mathrm{int}_X \mathrm{cl}_X(A \cup B)$ for any $A, B \in \mathscr{L}$, as noted in Section 2.) However, since we have both $A \subseteq A \vee B$ and $B \subseteq A \vee B$ we always have $A \cup B \subseteq A \vee B$.

The Stone–Čech compactification. Let X be a completely regular space. A compactification γX of X is a compact Hausdorff space γX containing X as a dense subspace. The *Stone–Čech compactification* βX of X is the compactification of X which is characterized among all compactifications of X by the following property: Every continuous $f : X \to K$, where K is a compact Hausdorff space, is continuously extendable over βX; denote by f_β this continuous extension of f. For a completely regular space the Stone–Čech compactification always exists. In what follows we will use the following properties of βX. (See Sections 3.5 and 3.6 of [7].)

- X is locally compact if and only if X is open in βX.
- Any open-closed subspace of X has open-closed closure in βX.
- If $X \subseteq T \subseteq \beta X$ then $\beta T = \beta X$.
- If X is normal then $\beta T = \mathrm{cl}_{\beta X} T$ for any closed subspace T of X.
- Disjoint zero-sets of X have disjoint closures in βX.

Part 1. General theory

This part is divided into two sections. Section 3 introduces the normed algebra $C_{00}^{\mathscr{I}}(X)$ and Section 4 introduces the Banach algebra $C_0^{\mathscr{I}}(X)$. Results of this part are stated and proved in the most general context; Part 2 will be subsequently devoted to the consideration of specific examples.

3. The normed algebra $C_{00}^{\mathscr{I}}(X)$

Let X be a space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L}, \subseteq)$, where $\mathscr{L} \subseteq \mathscr{P}(X)$. In this section we consider the subset $C_{00}^{\mathscr{I}}(X)$ of $C_b(X)$ consisting of those elements of $C_b(X)$ whose support has a null neighborhood in X. As we will see $C_{00}^{\mathscr{I}}(X)$ coincides with $C_{00}(X)$ if X is locally compact and $\mathscr{I}$ is the set of all subspaces of X with compact closure (and of course $\mathscr{L} = \mathscr{P}(X)$). We show that

$C_{00}^{\mathscr{I}}(X)$ is in general an ideal (and in particular a normed subalgebra) in $C_b(X)$. Furthermore, if X is completely regular, then $C_{00}^{\mathscr{I}}(X)$ is of empty hull if and only if X is locally null, and if so, then $C_{00}^{\mathscr{I}}(X)$ is unital if and only if $\mathscr{I}$ is non-proper. The main result of this section states that if X is normal and locally null then the normed algebra $C_{00}^{\mathscr{I}}(X)$ is isometrically isomorphic to $C_{00}(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y; this will be made possible through the crucial introduction of the subspace $\lambda_{\mathscr{I}}X$ of the Stone–Čech compactification βX of X. Furthermore, Y contains X densely, and is compact if and only if $C_{00}^{\mathscr{I}}(X)$ is unital. Our final result in this section which simplifies the expression of $C_{00}^{\mathscr{I}}(X)$ states that

$$C_{00}^{\mathscr{I}}(X)=\big\{f\in C_b(X):\mathrm{supp}(f)\text{ is null}\big\}$$

whenever X is Lindelöf and locally null, $\mathscr{I}$ is a σ-ideal and $\mathscr{L}$ contains the set $\mathscr{RC}(X)$ of all regular closed subspaces of X.

Results of this section will generalize those we already obtained in [16], [19] and [20]. We now proceed with the formal treatment of the subject.

Let X be a space and let A be a subspace of X. A subspace U of X is called a *neighborhood* of A in X if $A\subseteq \mathrm{int}_X U$.

Definition 3.1. Let X be a space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. Define

$$C_{00}^{\mathscr{I}}(X)=\big\{f\in C_b(X):\mathrm{supp}(f)\text{ has a null neighborhood in }X\big\}.$$

The following example justifies our use of the notation $C_{00}^{\mathscr{I}}(X)$.

Example 3.2. Let X be a locally compact space and let

$$\mathscr{I}=\{A\subseteq X:\mathrm{cl}_X A\text{ is compact}\}.$$

Trivially, $\mathscr{I}$ is an ideal in $(\mathscr{P}(X),\subseteq)$. As we see now, in this case we have

$$C_{00}^{\mathscr{I}}(X)=C_{00}(X).$$

This justifies our use of the notation $C_{00}^{\mathscr{I}}(X)$ here. Let $f\in C_b(X)$. It is obvious that if $\mathrm{supp}(f)$ has a null neighborhood U in X, then $\mathrm{supp}(f)$ is compact, as it is closed in $\mathrm{cl}_X U$ and the latter is so. Now, suppose that $\mathrm{supp}(f)$ is compact. For each $x\in X$ let V_x be an open neighborhood of x in X such that $\mathrm{cl}_X V_x$ is compact. The set $\{V_x:x\in X\}$ forms an open cover for $\mathrm{supp}(f)$. Therefore

$$\mathrm{supp}(f)\subseteq V_{x_1}\cup\cdots\cup V_{x_n}=V$$

for some $x_1,\dots,x_n\in X$. Clearly, V is a neighborhood of $\mathrm{supp}(f)$ in X, and it is null, as

$$\mathrm{cl}_X V=\mathrm{cl}_X V_{x_1}\cup\cdots\cup\mathrm{cl}_X V_{x_n},$$

(being the union of a finite number of compact subspaces) is compact.

Theorem 3.3. *Let X be a space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. Then $C_{00}^{\mathscr{I}}(X)$ is an ideal (in particular, a normed subalgebra) in $C_b(X)$.*

Proof. Note that $C_{00}^{\mathscr{I}}(X)$ is non-empty, as it contains the zero vector $\mathbf{0}$. (Observe that there always exists a null subset of X; this constitutes a neighborhood for $\emptyset=\mathrm{supp}(\mathbf{0})$ in X.) To show that $C_{00}^{\mathscr{I}}(X)$ is closed under addition, let $f,g\in$

$C_{00}^{\mathscr{I}}(X)$. Then, there exist null neighborhoods U and V of $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ in X, respectively. Note that $\mathrm{Coz}(f+g) \subseteq \mathrm{Coz}(f) \cup \mathrm{Coz}(g)$. Thus

$$\mathrm{supp}(f+g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g) \subseteq \mathrm{int}_X U \cup \mathrm{int}_X V \subseteq \mathrm{int}_X (U \cup V) \subseteq U \cup V \subseteq U \vee V.$$

Therefore $\mathrm{supp}(f+g)$ has a null neighborhood in X, namely $U \vee V$. Then $f+g \in C_{00}^{\mathscr{I}}(X)$. Next, let $f \in C_{00}^{\mathscr{I}}(X)$ and $g \in C_b(X)$. Note that $\mathrm{Coz}(fg) \subseteq \mathrm{Coz}(f)$. Thus $\mathrm{supp}(fg) \subseteq \mathrm{supp}(f)$. In particular, $\mathrm{supp}(fg)$ has a null neighborhoods in X, as $\mathrm{supp}(f)$ does. Therefore $fg \in C_{00}^{\mathscr{I}}(X)$. That $C_{00}^{\mathscr{I}}(X)$ is closed under scalar multiplication follows trivially. □

Remark 3.4. Following [9], we call an ideal H of $C_b(X)$ a *z-ideal* if $\mathrm{Z}(f) = \mathrm{Z}(h)$ where $f \in C_b(X)$ and $h \in H$ implies that $f \in H$. It is worth noting that in Theorem 3.3 the ideal $C_{00}^{\mathscr{I}}(X)$ is indeed a z-ideal; to see this let $f \in C_b(X)$ and $g \in C_{00}^{\mathscr{I}}(X)$ with $\mathrm{Z}(f) = \mathrm{Z}(g)$. Then $\mathrm{Coz}(f) = \mathrm{Coz}(g)$ and thus $\mathrm{supp}(f) = \mathrm{supp}(g)$. Therefore $\mathrm{supp}(f)$ has a null neighborhood in X, as $\mathrm{supp}(g)$ does. That is $f \in C_{00}^{\mathscr{I}}(X)$.

The subspace $\lambda_{\mathscr{I}} X$ of βX introduced below plays a crucial role in our study. The space $\lambda_{\mathscr{I}} X$ has been first considered in [14] (in a different context) to study certain classes of topological extensions. (See also [13], [15], [17] and [18].)

Definition 3.5. Let X be a completely regular space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L}, \subseteq)$, where $\mathscr{L} \subseteq \mathscr{P}(X)$. Define

$$\lambda_{\mathscr{I}} X = \bigcup \big\{ \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} C : C \in \mathrm{Coz}(X) \text{ and } \mathrm{cl}_X C \text{ has a null neighborhood in } X \big\},$$

considered as a subspace of βX.

Definition 3.6. Let X be a space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L}, \subseteq)$, where $\mathscr{L} \subseteq \mathscr{P}(X)$. Then X is called *locally null* (with respect to $\mathscr{I}$) if every point of X has a null neighborhood in X.

If X is a space and D is a dense subspace of X, then

$$\mathrm{cl}_X U = \mathrm{cl}_X (U \cap D)$$

for every open subspace U of X. We have the following simple observation.

Lemma 3.7. *Let X be a completely regular space. If $f : X \to [0,1]$ is continuous and $0 < r < 1$ then*

$$f_\beta^{-1}\big([0,r)\big) \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} f^{-1}\big([0,r)\big).$$

Proof. Note that

$$f_\beta^{-1}\big([0,r)\big) \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} f_\beta^{-1}\big([0,r)\big)$$

and

$$\mathrm{cl}_{\beta X} f_\beta^{-1}\big([0,r)\big) = \mathrm{cl}_{\beta X}\big(X \cap f_\beta^{-1}\big([0,r)\big)\big) = \mathrm{cl}_{\beta X} f^{-1}\big([0,r)\big).$$

□

Observe that if $f : \beta X \to [0,1]$ is continuous then $(f|_X)_\beta = f$, as they are both continuous and coincide on the dense subspace X of βX.

Let X be a space. For an ideal H of $C_b(X)$ the *hull* of H is defined to be

$$\mathfrak{h}(H) = \bigcap \big\{ \mathrm{Z}(h) : h \in H \big\};$$

the ideal H is said to be *of empty hull* (or *free*) if $\mathfrak{h}(H) = \emptyset$.

Theorem 3.8. *Let X be a completely regular space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. The following are equivalent:*

(1) $X\subseteq\lambda_{\mathscr{I}}X$.
(2) *X is locally null.*
(3) *$C^{\mathscr{I}}_{00}(X)$ is of empty hull.*

Proof. (1) *implies* (2). Let $x\in X$. Then $x\in\lambda_{\mathscr{I}}X$ and therefore $x\in\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}D$ for some $D\in\mathrm{Coz}(X)$ such that $\mathrm{cl}_X D$ has a null neighborhood V in X. But V is then a neighborhood of x in X as well, as $x\in\mathrm{cl}_{\beta X}D\cap X=\mathrm{cl}_X D$.

(2) *implies* (1). Let $x\in X$ and let U be a null neighborhood of x in X. Let $f:X\to[0,1]$ be continuous with $f(x)=0$ and $f|_{X\setminus\mathrm{int}_X U}\equiv\mathbf{1}$. Let

$$C=f^{-1}\big([0,1/2)\big)\in\mathrm{Coz}(X).$$

Then $\mathrm{cl}_X C\subseteq f^{-1}([0,1/2])$ and $f^{-1}([0,1/2])\subseteq\mathrm{int}_X U$. Thus U is a null neighborhood of $\mathrm{cl}_X C$ in X. Therefore $\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C\subseteq\lambda_{\mathscr{I}}X$. But then $x\in\lambda_{\mathscr{I}}X$, as $x\in f_\beta^{-1}([0,1/2))$ and $f_\beta^{-1}([0,1/2))\subseteq\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C$ by Lemma 3.7.

(2) *implies* (3). Note that $C^{\mathscr{I}}_{00}(X)$ is an ideal in $C_b(X)$ by Theorem 3.3. Let $x\in X$ and let V be a null neighborhood of x in X. Let W be an open neighborhood of x in X with $\mathrm{cl}_X W\subseteq\mathrm{int}_X V$. Let $g:X\to[0,1]$ be continuous with $g(x)=1$ and $g|_{X\setminus W}\equiv\mathbf{0}$. Then $\mathrm{supp}(g)\subseteq\mathrm{cl}_X W$, as $\mathrm{Coz}(g)\subseteq W$. Thus V is a null neighborhood of $\mathrm{supp}(g)$ in X. Therefore $g\in C^{\mathscr{I}}_{00}(X)$.

(3) *implies* (2). Let $x\in X$. Then $x\notin\mathrm{Z}(h)$ for some $h\in C^{\mathscr{I}}_{00}(X)$. Since $\mathrm{supp}(h)$ has a null neighborhood in X and $x\in\mathrm{supp}(h)$, as $x\in\mathrm{Coz}(h)$, it then follows that x has a null neighborhood in X. □

Lemma 3.9. *Let X be a completely regular space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. For any subspace A of X, if $\mathrm{cl}_{\beta X}A\subseteq\lambda_{\mathscr{I}}X$ then $\mathrm{cl}_X A$ has a null neighborhood in X.*

Proof. By compactness, we have

$$\mathrm{cl}_{\beta X}A\subseteq\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C_1\cup\cdots\cup\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C_n \tag{3.1}$$

for some $C_1,\ldots,C_n\in\mathrm{Coz}(X)$ such that each $\mathrm{cl}_X C_i$, where $i=1,\ldots,n$, has a null neighborhood U_i in X. Intersecting both sides of (3.1) with X we have

$$\mathrm{cl}_X A\subseteq\mathrm{cl}_X C_1\cup\cdots\cup\mathrm{cl}_X C_n.$$

Since

$$\mathrm{cl}_X C_1\cup\cdots\cup\mathrm{cl}_X C_n\subseteq U_1\cup\cdots\cup U_n\subseteq U_1\vee\cdots\vee U_n=W$$

it follows that W is a null neighborhood of $\mathrm{cl}_X A$ in X. □

Theorem 3.10. *Let X be a completely regular space locally null with respect to an ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. The following are equivalent:*

(1) *$\lambda_{\mathscr{I}}X$ is compact.*
(2) *$\mathscr{I}$ is non-proper.*
(3) *$C^{\mathscr{I}}_{00}(X)$ is unital.*

Proof. (1) *implies* (2). Note that $X\subseteq\lambda_{\mathscr{I}}X$ by Theorem 3.8, as X is locally null. Since $\lambda_{\mathscr{I}}X$ is compact we have $\mathrm{cl}_{\beta X}X\subseteq\lambda_{\mathscr{I}}X$. Therefore X is null by Lemma 3.9.

(2) *implies* (3). If X is null then the function $\mathbf{1}$ (defined to be identically 1 on the whole X) is the unit element of $C^{\mathscr{I}}_{00}(X)$.

(3) *implies* (1). Suppose that $C_{00}^{\mathscr{I}}(X)$ has a unit element u. Let $x\in X$. Let U_x be a null neighborhood of x in X and let V_x be an open neighborhoods of x in X such that $\mathrm{cl}_X V_x\subseteq \mathrm{int}_X U_x$. Let $f_x:X\to[0,1]$ be continuous and such that $f_x(x)=1$ and $f_x|_{X\setminus V_x}\equiv\mathbf{0}$. Then U_x is a neighborhood of $\mathrm{supp}(f_x)$ in X, as $\mathrm{supp}(f_x)\subseteq \mathrm{cl}_X V_x$. Therefore $f_x\in C_{00}^{\mathscr{I}}(X)$. We have

$$u(x)=u(x)f_x(x)=f_x(x)=1.$$

Thus $u=\mathbf{1}$ and therefore $X=\mathrm{supp}(u)$ is null. Since $X\in\mathrm{Coz}(X)$ trivially, it follows that $\lambda_{\mathscr{I}}X=\beta X$ is compact. □

Definition 3.11. Let X be a completely regular space locally null with respect to an ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. For any $f\in C_b(X)$ denote $f_\lambda=f_\beta|_{\lambda_{\mathscr{I}}X}$.

Observe that by Theorem 3.8 the function f_λ extends f.

Lemma 3.12. *Let X be a normal space locally null with respect to an ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. For any $f\in C_b(X)$ the following are equivalent:*

(1) $f\in C_{00}^{\mathscr{I}}(X)$.
(2) $f_\lambda\in C_{00}(\lambda_{\mathscr{I}}X)$.

Proof. Note that $X\subseteq\lambda_{\mathscr{I}}X$ by Theorem 3.8, as X is locally null.

(1) *implies* (2). Let U be a null neighborhood of $\mathrm{supp}(f)$ in X. Thus $\mathrm{supp}(f)\subseteq \mathrm{int}_X U$. Since X is normal, by the Urysohn Lemma, there exists a continuous $g:X\to[0,1]$ such that

$$g|_{\mathrm{supp}(f)}\equiv\mathbf{0}\quad\text{ and }\quad g|_{X\setminus \mathrm{int}_X U}\equiv\mathbf{1}.$$

Let

$$C=g^{-1}\big([0,1/2)\big)\in\mathrm{Coz}(X).$$

Then $\mathrm{cl}_X C$ has a null neighborhood in X, namely U, as

$$\mathrm{cl}_X C\subseteq g^{-1}\big([0,1/2]\big)\subseteq \mathrm{int}_X U.$$

Therefore $\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C\subseteq\lambda_{\mathscr{I}}X$. But $g_\beta^{-1}([0,1/2))\subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C$ by Lemma 3.7, and thus

$$\mathrm{cl}_{\beta X}\mathrm{Coz}(f)\subseteq \mathrm{Z}(g_\beta)\subseteq g_\beta^{-1}\big([0,1/2)\big)\subseteq\lambda_{\mathscr{I}}X.$$

This implies that

$$\begin{aligned}\mathrm{supp}(f_\lambda)=\mathrm{cl}_{\lambda_{\mathscr{I}}X}\mathrm{Coz}(f_\lambda) &= \mathrm{cl}_{\lambda_{\mathscr{I}}X}\big(X\cap\mathrm{Coz}(f_\lambda)\big)\\ &= \mathrm{cl}_{\lambda_{\mathscr{I}}X}\mathrm{Coz}(f)=\lambda_{\mathscr{I}}X\cap\mathrm{cl}_{\beta X}\mathrm{Coz}(f)=\mathrm{cl}_{\beta X}\mathrm{Coz}(f)\end{aligned}$$

is compact, as it is closed in βX.

(2) *implies* (1). Note that $\mathrm{cl}_{\beta X}(\mathrm{supp}(f))\subseteq\mathrm{supp}(f_\lambda)$, as $\mathrm{supp}(f)\subseteq\mathrm{supp}(f_\lambda)$ and the latter is compact. Thus $\mathrm{cl}_{\beta X}(\mathrm{supp}(f))\subseteq\lambda_{\mathscr{I}}X$. By Lemma 3.9 it follows that $\mathrm{supp}(f)$ has a null neighborhood in X. □

A version of the classical Banach–Stone Theorem (see Theorem 7.1 of [4]) states that for any locally compact Hausdorff spaces X and Y, the rings $C_{00}(X)$ and $C_{00}(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic. (See [3].) This will be used in the proof of the following theorem.

Theorem 3.13. *Let X be a normal space locally null with respect to an ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. Then the normed algebra $C_{00}^{\mathscr{I}}(X)$ is isometrically isomorphic to $C_{00}(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y, namely $Y=\lambda_{\mathscr{I}}X$. Furthermore,*

1. *$C_{00}^{\mathscr{I}}(X)$ is of empty hull.*
2. *X is dense in Y.*
3. *$C_{00}^{\mathscr{I}}(X)$ is unital if and only if $\mathscr{I}$ is non-proper if and only if Y is compact.*

Proof. Observe that $C_{00}^{\mathscr{I}}(X)$ is a normed subalgebra of $C_b(X)$ by Theorem 3.3. Define

$$\psi : C_{00}^{\mathscr{I}}(X)\to C_{00}(\lambda_{\mathscr{I}}X)$$

by

$$\psi(f)=f_\lambda$$

for any $f\in C_{00}^{\mathscr{I}}(X)$. By Lemma 3.12 the function ψ is well-defined. It is clear that ψ is an algebra homomorphism and that it is injective. (Again, note that $X\subseteq\lambda_{\mathscr{I}}X$, and use the fact that any two scalar-valued continuous functions on $\lambda_{\mathscr{I}}X$ coincide, provided that they agree on the dense subspace X of $\lambda_{\mathscr{I}}X$.) To show that ψ is surjective, let $g\in C_{00}(\lambda_{\mathscr{I}}X)$. Then $(g|_X)_\lambda=g$ and thus $g|_X\in C_{00}^{\mathscr{I}}(X)$ by Lemma 3.12. Note that $\psi(g|_X)=g$. To show that ψ is an isometry, let $h\in C_{00}^{\mathscr{I}}(X)$. Then

$$|h_\lambda|(\lambda_{\mathscr{I}}X)=|h_\lambda|(\mathrm{cl}_{\lambda_{\mathscr{I}}X}X)\subseteq\mathrm{cl}_{\mathbb{R}}\big(|h_\lambda|(X)\big)=\mathrm{cl}_{\mathbb{R}}\big(|h|(X)\big)\subseteq\big[0,\|h\|\big]$$

which yields $\|h_\lambda\|\le\|h\|$. That $\|h\|\le\|h_\lambda\|$ is clear, as h_λ extends h.

Note that $\lambda_{\mathscr{I}}X$ is locally compact, as it is open in the compact Hausdorff space βX.

The uniqueness of $\lambda_{\mathscr{I}}X$ follows from the fact that for any locally compact Hausdorff space T the ring $C_{00}(T)$ determines the topology of T.

(1). This follows from Theorem 3.8.

(2). By Theorem 3.8 we have $X\subseteq\lambda_{\mathscr{I}}X$. That X is dense in $\lambda_{\mathscr{I}}X$ is then obvious.

(3). This follows from Theorem 3.10. □

Under certain conditions the representation given for $C_{00}^{\mathscr{I}}(X)$ simplifies. This is the context of our next result. First, we need a lemma.

Lemma 3.14. *Let X be a Lindelöf space locally null with respect to a σ-ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. Then the closure in X of each null subset of X has a null neighborhood in X.*

Proof. Let A be a null subset of X. Since X is locally null, there exists a null neighborhood U_x of x in X for each $x\in X$. Since

$$\mathrm{cl}_X A\subseteq\bigcup_{x\in X}\mathrm{int}_X U_x$$

and $\mathrm{cl}_X A$ is Lindelöf, as it is closed in X and X is so, we have

$$\mathrm{cl}_X A\subseteq\bigcup_{n=1}^\infty\mathrm{int}_X U_{x_n}.$$

But then, since $\mathscr{I}$ is a σ-ideal, we can write

$$\bigcup_{n=1}^\infty\mathrm{int}_X U_{x_n}\subseteq\bigcup_{n=1}^\infty U_{x_n}\subseteq\bigvee_{n=1}^\infty U_{x_n}=W.$$

That is, $\mathrm{cl}_X A$ has a null neighborhood in X, namely W. □

Recall that $\mathscr{RC}(X)$ denotes the set of all regular closed subspaces of a space X. (See Section 2)

Theorem 3.15. *Let X be a Lindelöf space locally null with respect to a σ-ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. If $\mathscr{RC}(X)\subseteq\mathscr{L}$ then*

$$C_{00}^{\mathscr{I}}(X)=\big\{f\in C_b(X):\mathrm{supp}(f)\ \textit{is null}\big\}.$$

Proof. Let $f\in C_{00}^{\mathscr{I}}(X)$. Then $\mathrm{supp}(f)\subseteq U$ for some null U. Since $\mathrm{supp}(f)\in\mathscr{L}$, as $\mathrm{supp}(f)$ is regular closed in X, it follows that $\mathrm{supp}(f)$ is null. The converse trivially follows from Lemma 3.14. □

4. The Banach algebra $C_0^{\mathscr{I}}(X)$

Let X be a space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. In this section we consider the subset $C_0^{\mathscr{I}}(X)$ of $C_b(X)$ consisting of those elements f of $C_b(X)$ such that $|f|^{-1}([1/n,\infty))$ has a null neighborhood in X for each positive integer n. As we will see, $C_0^{\mathscr{I}}(X)$ coincides with $C_0(X)$ if X is locally compact and $\mathscr{I}$ is the set of all subspaces of X with compact closure (and of course $\mathscr{L}=\mathscr{P}(X)$). Under certain conditions the expression of $C_0^{\mathscr{I}}(X)$ simplifies, this includes the case when $\mathscr{L}$ contains the set $\mathscr{Z}(X)$ of all zero-sets of X, in which case

$$C_0^{\mathscr{I}}(X)=\big\{f\in C_b(X):|f|^{-1}\big([1/n,\infty)\big)\text{ is null for each }n\big\}.$$

We show that $C_0^{\mathscr{I}}(X)$ is in general a closed ideal (and in particular, a Banach subalgebra) in $C_b(X)$ containing $C_{00}^{\mathscr{I}}(X)$. Furthermore, if X is completely regular, then $C_0^{\mathscr{I}}(X)$ is of empty hull if and only if X is locally null, and if so, then $C_0^{\mathscr{I}}(X)$ is unital if and only if $\mathscr{I}$ is non-proper. The main result of this section states that if X is normal and locally null then the Banach algebra $C_0^{\mathscr{I}}(X)$ is isometrically isomorphic to $C_0(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y. Furthermore, $C_0^{\mathscr{I}}(X)$ contains $C_{00}^{\mathscr{I}}(X)$ densely, Y contains X densely, and Y is compact if and only if $C_0^{\mathscr{I}}(X)$ is unital. Also, if X is moreover Lindelöf and $\mathscr{I}$ is a σ-ideal then Y is countably compact and $C_0^{\mathscr{I}}(X)=C_{00}^{\mathscr{I}}(X)$.

We now proceed with the formal treatment of the subject.

Definition 4.1. Let X be a space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. Define

$$C_0^{\mathscr{I}}(X)=\big\{f\in C_b(X):|f|^{-1}\big([1/n,\infty)\big)\text{ has a null neighborhood in }X\text{ for each }n\big\}.$$

The following is to justify our use of the notation $C_0^{\mathscr{I}}(X)$.

Example 4.2. Let X be a locally compact space. Consider the ideal

$$\mathscr{I}=\{A\subseteq X:\mathrm{cl}_X A\text{ is compact}\}.$$

of $(\mathscr{P}(X),\subseteq)$. Then, an argument analogous to the one given in Example 3.2 shows that

$$C_0^{\mathscr{I}}(X)=C_0(X).$$

Under certain conditions the representation of $C_0^{\mathscr{I}}(X)$ given in Definition 4.1 simplifies. This is the context of the next few results. Recall that $\mathscr{Z}(X)$ denotes the set of all zero-sets of a space X. (See Section 2.)

Proposition 4.3. *Let X be a space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. If $\mathscr{Z}(X)\subseteq\mathscr{L}$ then*

$$C_0^{\mathscr{I}}(X)=\big\{f\in C_b(X):|f|^{-1}\big([1/n,\infty)\big)\text{ is null for each } n\big\}.$$

Proof. Let $f\in C_0^{\mathscr{I}}(X)$. Let n be a positive integer. Then $|f|^{-1}([1/n,\infty))$ is contained in a null subset of X, and is therefore null itself. (Note that $|f|^{-1}([1/n,\infty))$ is contained in $\mathscr{L}$, as it is contained in $\mathscr{Z}(X)$; to see the latter, let

$$g=\max\Big\{\mathbf{0},\frac{\mathbf{1}}{\mathbf{n}}-|f|\Big\}$$

and observe that $\mathrm{Z}(g)=|f|^{-1}([1/n,\infty))$.)

Next, let $f\in C_b(X)$ such that $|f|^{-1}([1/n,\infty))$ is null for each positive integer n. Since

$$|f|^{-1}\big([1/n,\infty)\big)\subseteq|f|^{-1}\Big(\Big(\frac{1}{n+1},\infty\Big)\Big)\subseteq|f|^{-1}\Big(\Big[\frac{1}{n+1},\infty\Big)\Big),$$

it follows that the latter is a null neighborhood of $|f|^{-1}([1/n,\infty))$ in X, for each positive integer n. That is $f\in C_0^{\mathscr{I}}(X)$. $\square$

Proposition 4.4. *Let X be a space and let $\mathscr{I}$ be a σ-ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. If $\mathrm{Coz}(X)\subseteq\mathscr{L}$ then*

$$C_0^{\mathscr{I}}(X)=\big\{f\in C_b(X):\mathrm{Coz}(f)\text{ is null}\big\}.$$

Proof. Let $f\in C_0^{\mathscr{I}}(X)$. Let n be a positive integer. Then $|f|^{-1}([1/n,\infty))$ has a null neighborhood U_n in X. We have

$$\mathrm{Coz}(f)=\bigcup_{n=1}^{\infty}|f|^{-1}\big([1/n,\infty)\big)\subseteq\bigcup_{n=1}^{\infty}\mathrm{int}_X U_n\subseteq\bigcup_{n=1}^{\infty}U_n\subseteq\bigvee_{n=1}^{\infty}U_n=V.$$

Since by our assumption $\mathrm{Coz}(f)\in\mathscr{L}$ (and V is null) it follows that $\mathrm{Coz}(f)$ is null.

Next, note that if $\mathrm{Coz}(f)$ is null, where $f\in C_b(X)$, then $|f|^{-1}([1/n,\infty))$ has a null neighborhood in X for each positive integer n, namely $\mathrm{Coz}(f)$ itself. Thus $f\in C_0^{\mathscr{I}}(X)$. $\square$

Theorem 4.5. *Let X be a space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. Then $C_0^{\mathscr{I}}(X)$ is a closed ideal (in particular, a Banach subalgebra) in the normed algebra $C_b(X)$. Furthermore,*

$$C_{00}^{\mathscr{I}}(X)\subseteq C_0^{\mathscr{I}}(X).$$

Proof. We start from the last statement. Let $f\in C_{00}^{\mathscr{I}}(X)$. Let n be a positive integer. Then $|f|^{-1}([1/n,\infty))$ has a null neighborhood in X, as it is contained in $\mathrm{supp}(f)$ and the latter does. Thus $f\in C_0^{\mathscr{I}}(X)$.

Next, we show that $C_0^{\mathscr{I}}(X)$ is an ideal in the algebra $C_b(X)$. To show that $C_0^{\mathscr{I}}(X)$ is closed under addition, let $f,g\in C_0^{\mathscr{I}}(X)$. Let n be a positive integer. There exist null neighborhoods U and V of $|f|^{-1}([1/(2n),\infty))$ and $|g|^{-1}([1/(2n),\infty))$ in X, respectively. Then

$$\begin{aligned}|f+g|^{-1}\big([1/n,\infty)\big)&\subseteq|f|^{-1}\Big(\Big[\frac{1}{2n},\infty\Big)\Big)\cup|g|^{-1}\Big(\Big[\frac{1}{2n},\infty\Big)\Big)\\&\subseteq\mathrm{int}_X U\cup\mathrm{int}_X V\subseteq U\cup V\subseteq U\vee V.\end{aligned}$$

Thus $|f+g|^{-1}([1/n,\infty))$ has a null neighborhood in X, namely $U \vee V$. Therefore $f+g \in C_0^{\mathscr{I}}(X)$. Next, let $f \in C_0^{\mathscr{I}}(X)$ and $g \in C_b(X)$. Let m be a positive integer such that $|g(x)| \leq m$ for each $x \in X$. Since

$$|fg|^{-1}\big([1/n,\infty)\big) \subseteq |f|^{-1}\Big(\Big[\frac{1}{mn},\infty\Big)\Big),$$

and $|f|^{-1}([1/(mn),\infty))$ has a null neighborhoods in X it follows that $|fg|^{-1}([1/n,\infty))$ has a null neighborhoods in X. Therefore $fg \in C_0^{\mathscr{I}}(X)$. That $C_0^{\mathscr{I}}(X)$ is closed under scalar multiplication follows analogously.

Finally, we show that $C_0^{\mathscr{I}}(X)$ is closed in $C_b(X)$. Let f be in the closure in $C_b(X)$ of $C_0^{\mathscr{I}}(X)$. Let n be a positive integer. There exists some $g \in C_0^{\mathscr{I}}(X)$ with $\|f-g\| < 1/(2n)$. If $t \in |f|^{-1}([1/n,\infty))$ then

$$\frac{1}{n} \leq \big|f(t)\big| \leq \big|f(t)-g(t)\big| + \big|g(t)\big| \leq \|f-g\| + \big|g(t)\big| \leq \frac{1}{2n} + \big|g(t)\big|$$

and thus $|g(t)| \geq 1/(2n)$. That is $t \in |g|^{-1}([1/(2n),\infty))$. Therefore

$$|f|^{-1}\big([1/n,\infty)\big) \subseteq |g|^{-1}\Big(\Big[\frac{1}{2n},\infty\Big)\Big).$$

Since the latter has a null neighborhood in X, so does $|f|^{-1}([1/n,\infty))$. Thus $f \in C_0^{\mathscr{I}}(X)$. □

Theorem 4.6. *Let X be a completely regular space and let $\mathscr{I}$ be an ideal in an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L} \subseteq \mathscr{P}(X)$. The following are equivalent:*

(1) *$X \subseteq \lambda_{\mathscr{I}} X$.*
(2) *X is locally null.*
(3) *$C_0^{\mathscr{I}}(X)$ is of empty hull.*
(4) *$C_{00}^{\mathscr{I}}(X)$ is of empty hull.*

Proof. The equivalence of (1), (2) and (4) follows from Theorem 3.8.

(4) *implies* (3). Note that $C_0^{\mathscr{I}}(X)$ contains $C_{00}^{\mathscr{I}}(X)$ by Theorem 4.5. Thus $C_0^{\mathscr{I}}(X)$ is of empty hull if $C_{00}^{\mathscr{I}}(X)$ is so.

(3) *implies* (2). Let $x \in X$. Then $x \notin \mathrm{Z}(f)$ for some $f \in C_0^{\mathscr{I}}(X)$. That is $|f(x)| > 0$. Let n be a positive integer such that $|f(x)| \geq 1/n$. Then $x \in |f|^{-1}([1/n,\infty))$. Since $|f|^{-1}([1/n,\infty))$ has a null neighborhood in X it follows that x has a null neighborhood in X. □

Theorem 4.7. *Let X be a completely regular space locally null with respect to an ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L} \subseteq \mathscr{P}(X)$. The following are equivalent:*

(1) *$\lambda_{\mathscr{I}} X$ is compact.*
(2) *$\mathscr{I}$ is non-proper.*
(3) *$C_0^{\mathscr{I}}(X)$ is unital.*
(4) *$C_{00}^{\mathscr{I}}(X)$ is unital.*

Proof. The equivalence of (1), (2) and (4) follows from Theorem 3.10.

(2) *implies* (3). If X is null then the function $\mathbf{1}$ is the unit element of $C_0^{\mathscr{I}}(X)$.

(3) *implies* (2). Suppose that $C_0^{\mathscr{I}}(X)$ has a unit element u. Let U_x, V_x and f_x be as defined in the proof of Theorem 3.10. Note that $f_x \in C_0^{\mathscr{I}}(X)$, as $f_x \in C_{00}^{\mathscr{I}}(X)$ and $C_{00}^{\mathscr{I}}(X) \subseteq C_0^{\mathscr{I}}(X)$ by Theorem 4.5. Arguing as in the proof of Theorem 3.10 we have $u = \mathbf{1}$. This implies that $X = u^{-1}([1,\infty))$ is null. □

Lemma 4.8. *Let X be a normal space locally null with respect to an ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. For any $f\in C_b(X)$ the following are equivalent:*

(1) $f\in C_0^{\mathscr{I}}(X)$.
(2) $f_\lambda\in C_0(\lambda_{\mathscr{I}}X)$.

Proof. Note that $X\subseteq\lambda_{\mathscr{I}}X$ by Theorem 3.8, as X is locally null.

(1) *implies* (2). Let k be a positive integer. Let U_k be a null neighborhood of $|f|^{-1}([1/k,\infty))$ in X. Then $|f|^{-1}([1/k,\infty))\subseteq\mathrm{int}_X U_k$. Since X is normal, by the Urysohn Lemma, there exists a continuous $g_k:X\to[0,1]$ such that

$$g_k|_{|f|^{-1}([1/k,\infty))}\equiv\mathbf{0}\quad\text{and}\quad g_k|_{X\setminus\mathrm{int}_X U_k}\equiv\mathbf{1}.$$

Let

$$C_k=g_k^{-1}\big([0,1/2)\big)\in\mathrm{Coz}(X).$$

Then $\mathrm{cl}_X C_k$ has a null neighborhood in X, namely U_k, as

$$\mathrm{cl}_X C_k\subseteq g_k^{-1}\big([0,1/2]\big)\subseteq\mathrm{int}_X U_k.$$

Therefore $\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C_k\subseteq\lambda_{\mathscr{I}}X$. Arguing as in the proof of Lemma 3.7 we have

$$|f_\beta|^{-1}\big((1/k,\infty)\big)\subseteq\mathrm{cl}_{\beta X}\big(|f|^{-1}\big((1/k,\infty)\big)\big).$$

Since

$$\mathrm{cl}_{\beta X}\big(|f|^{-1}\big((1/k,\infty)\big)\big)\subseteq\mathrm{cl}_{\beta X}\big(\mathrm{Z}(g_k)\big)\subseteq\mathrm{Z}\big((g_k)_\beta\big)\subseteq(g_k)_\beta^{-1}\big([0,1/2)\big)$$

and

$$(g_k)_\beta^{-1}\big([0,1/2)\big)\subseteq\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C_k$$

by Lemma 3.7, it follows that

$$|f_\beta|^{-1}\big((1/k,\infty)\big)\subseteq\lambda_{\mathscr{I}}X. \tag{4.1}$$

Now, let n be a positive integer. Using (4.1), we have

$$|f_\beta|^{-1}\big([1/n,\infty)\big)\subseteq|f_\beta|^{-1}\Big(\Big(\frac{1}{n+1},\infty\Big)\Big)\subseteq\lambda_{\mathscr{I}}X.$$

Therefore

$$|f_\lambda|^{-1}\big([1/n,\infty)\big)=\lambda_{\mathscr{I}}X\cap|f_\beta|^{-1}\big([1/n,\infty)\big)=|f_\beta|^{-1}\big([1/n,\infty)\big)$$

is compact, as it is closed in βX.

(2) *implies* (1). Let n be a positive integer. Since $|f_\lambda|^{-1}([1/n,\infty))$ contains $|f|^{-1}([1/n,\infty))$ and it is compact, we have

$$\mathrm{cl}_{\beta X}\big(|f|^{-1}\big([1/n,\infty)\big)\big)\subseteq|f_\lambda|^{-1}\big([1/n,\infty)\big)\subseteq\lambda_{\mathscr{I}}X.$$

But then $|f|^{-1}([1/n,\infty))$ has a null neighborhood in X by Lemma 3.9. □

There is a version of the classical Banach–Stone Theorem which states that for any locally compact Hausdorff spaces X and Y, the rings $C_0(X)$ and $C_0(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic. (See [3].) This will be used in the proof of the following theorem.

Theorem 4.9. *Let X be a normal space locally null with respect to an ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L},\subseteq)$, where $\mathscr{L}\subseteq\mathscr{P}(X)$. Then the Banach algebra $C_0^{\mathscr{I}}(X)$ is isometrically isomorphic to $C_0(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y, namely $Y=\lambda_{\mathscr{I}}X$. Furthermore,*

(1) *$C_0^{\mathscr{I}}(X)$ is of empty hull.*

(2) *X is dense in Y.*
(3) *$C_{00}^{\mathscr{I}}(X)$ is dense in $C_0^{\mathscr{I}}(X)$.*
(4) *$C_0^{\mathscr{I}}(X)$ is unital if and only if $\mathscr{I}$ is non-proper if and only if Y is compact.*

Proof. Observe that $C_0^{\mathscr{I}}(X)$ is a Banach subalgebra of $C_b(X)$ by Theorem 4.5. Define

$$\phi : C_0^{\mathscr{I}}(X) \to C_0(\lambda_{\mathscr{I}} X)$$

by

$$\phi(f) = f_\lambda$$

for any $f \in C_0^{\mathscr{I}}(X)$. By Lemma 4.8 the function ϕ is well-defined, and arguing as in the proof of Theorem 3.13, it follows that ϕ is an isometric algebra isomorphism.

Note that $\lambda_{\mathscr{I}} X$ is locally compact, as it is open in the compact Hausdorff space βX, and $\lambda_{\mathscr{I}} X$ contains X (as a dense subspace) by Theorem 3.8.

The uniqueness of $\lambda_{\mathscr{I}} X$ follows from the fact that the topology of any locally compact Hausdorff space T is determined by the algebraic structure of the ring $C_0(T)$.

(1). This follows from Theorem 4.6.

(2). By Theorem 4.6 we have $X \subseteq \lambda_{\mathscr{I}} X$. That X is dense in $\lambda_{\mathscr{I}} X$ is then obvious.

(3). Let ϕ be as defined in the above and let $\psi = \phi|_{C_{00}^{\mathscr{I}}(X)}$. Then

$$\psi : C_{00}^{\mathscr{I}}(X) \to C_{00}(\lambda_{\mathscr{I}} X),$$

and ψ is surjective by the proof of Theorem 3.13. The result now follows from the well known fact that $C_{00}(T)$ is dense in $C_0(T)$ for any locally compact Hausdorff space T.

(4). This follows from Theorem 4.7. □

Remark 4.10. In Theorem 4.9, assuming that $C_0^{\mathscr{I}}(X)$ is a Banach algebra, it follows from the Commutative Gelfand–Naimark Theorem that $C_0^{\mathscr{I}}(X)$ is isometrically isomorphic to $C_0(Y)$ for some locally compact Hausdorff space Y. Our approach here, apart from its independence, has the advantage of giving extra information, about either the Banach algebra $C_0^{\mathscr{I}}(X)$ or the space Y, not generally expected to be deducible from the standard Gelfand Theory. This fact is particularly highlighted in the second part of the article in which we consider specific examples of the space X or the ideal $\mathscr{I}$.

Let X be a locally compact non-compact Hausdorff space. It is known that $C_0(X) = C_{00}(X)$ if and only if every σ-compact subspace of X is contained in a compact subspace of X. (See Problem 7G.2 of [9].) In particular, $C_0(X) = C_{00}(X)$ implies that X is countably compact. (Recall that a space X is countably compact if and only if each countably infinite subspace of X has an accumulation point in X; see Theorem 3.10.3 of [7].) This will be used in the proof of the following result which examines conditions under which $C_{00}^{\mathscr{I}}(X)$ and $C_0^{\mathscr{I}}(X)$ coincide.

Theorem 4.11. *Let X be a regular Lindelöf space locally null with respect to a σ-ideal $\mathscr{I}$ of an upper semi-lattice $(\mathscr{L}, \subseteq)$, where $\mathscr{L} \subseteq \mathscr{P}(X)$. Then*

(1) *$C_0^{\mathscr{I}}(X) = C_{00}^{\mathscr{I}}(X)$.*
(2) *$C_0(\lambda_{\mathscr{I}} X) = C_{00}(\lambda_{\mathscr{I}} X)$.*
(3) *$\lambda_{\mathscr{I}} X$ is countably compact.*

Proof. We show that every σ-compact subspace of $\lambda_{\mathscr{I}} X$ is contained in a compact subspace of $\lambda_{\mathscr{I}} X$; this will show (2). (Note that $\lambda_{\mathscr{I}} X$ is locally compact, as it is open in βX.) In particular, since every countable set is σ-compact, this will prove that every countably infinite subspace of $\lambda_{\mathscr{I}} X$ has an accumulation point in $\lambda_{\mathscr{I}} X$, that is, $\lambda_{\mathscr{I}} X$ is countably compact, and therefore (3) holds as well.

Let A be a σ-compact subspace of $\lambda_{\mathscr{I}} X$. Then $A = A_1 \cup A_2 \cup \cdots$ where each $A_1, A_2, \ldots$ is compact. For each positive integer n, by compactness of A_n we have

$$A_n \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} C_1^n \cup \cdots \cup \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} C_{k_n}^n \tag{4.2}$$

for some $C_1^n, \ldots, C_{k_n}^n \in \mathrm{Coz}(X)$ such that each $\mathrm{cl}_X C_i^n$, where $i = 1, \ldots, k_n$, has a null neighborhood U_i^n in X. Let

$$V = \bigvee_{n=1}^{\infty} \bigvee_{i=1}^{k_n} U_i^n.$$

Then V is null, as we are assuming that $\mathscr{I}$ is a σ-ideal. By Lemma 3.14 there exists a null neighborhood W of $\mathrm{cl}_X V$ in X. Note that every regular Lindelöf space is normal. Now, since X is normal, by the Urysohn Lemma there exists a continuous $f : X \to [0, 1]$ with

$$f|_{\mathrm{cl}_X V} \equiv \mathbf{0} \quad \text{and} \quad f|_{X \setminus \mathrm{int}_X W} \equiv \mathbf{1}.$$

We prove that $\mathrm{Z}(f_\beta)$ is the desired compact subspace of $\lambda_{\mathscr{I}} X$ which contains A. Let

$$C = f^{-1}\big([0, 1/2)\big) \in \mathrm{Coz}(X).$$

Note that $\mathrm{cl}_X C$ has a null neighborhood in X, namely W, as

$$\mathrm{cl}_X C \subseteq f^{-1}\big([0, 1/2]\big) \subseteq \mathrm{int}_X W.$$

Therefore $\mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} C \subseteq \lambda_{\mathscr{I}} X$. But $f_\beta^{-1}([0, 1/2)) \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} C$ by Lemma 3.7. Since $\mathrm{Z}(f_\beta) \subseteq f_\beta^{-1}([0, 1/2))$ we then have $\mathrm{Z}(f_\beta) \subseteq \lambda_{\mathscr{I}} X$. Observe that $\mathrm{Z}(f_\beta)$ is closed in βX and is therefore compact. Now, let $n = 1, 2, \ldots$ and $i = 1, \ldots, k_n$. Since $C_i^n \subseteq U_i^n \subseteq V$, by the way we have defined f it follows that

$$\mathrm{cl}_{\beta X} C_i^n \subseteq \mathrm{cl}_{\beta X}\big(\mathrm{Z}(f)\big) \subseteq \mathrm{Z}(f_\beta). \tag{4.3}$$

From (4.2) and (4.3), we then have $A_n \subseteq \mathrm{Z}(f_\beta)$ for each positive integer n. Therefore $\mathrm{Z}(f_\beta)$ contains A.

(1). As we have seen in the proofs of Theorems 3.13 and 4.9 the function

$$\psi : C_0^{\mathscr{I}}(X) \to C_0(\lambda_{\mathscr{I}} X)$$

defined by

$$\psi(f) = f_\lambda$$

for any $f \in C_0^{\mathscr{I}}(X)$ is a bijection with

$$\psi\big(C_{00}^{\mathscr{I}}(X)\big) = C_{00}(\lambda_{\mathscr{I}} X).$$

The equivalence of (1) and (2) is now immediate. $\square$

Remark 4.12. As we have seen in Theorem 4.11, if $\mathscr{I}$ is a proper σ-ideal then $\lambda_{\mathscr{I}} X$ is a non-compact countably compact space with $C_0(\lambda_{\mathscr{I}} X) = C_{00}(\lambda_{\mathscr{I}} X)$. In particular, if $\mathfrak{P}$ is a topological property such that

$$\mathfrak{P} + \text{countable compactness} \to \text{compactness}$$

then $\lambda_{\mathscr{I}} X$ is non-$\mathfrak{P}$ either. The list of such topological properties is quite long and includes topological properties such as the Lindelöf property, paracompactness, realcompactness, metacompactness, subparacompactness, submetacompactness (or θ-refinability), the meta-Lindelöf property, the submeta-Lindelöf property (or $\delta\theta$-refinability), weak submetacompactness (or weak θ-refinability) and the weak submeta-Lindelöf property (or weak $\delta\theta$-refinability) among others. (See Parts 6.1 and 6.2 of [27].)

Part 2. Examples

In this part we study specific examples of either $C_{00}^{\mathscr{I}}(X)$ or $C_0^{\mathscr{I}}(X)$. We will see how this specification, either of the space X or the ideal $\mathscr{I}$, enables us to study $C_{00}^{\mathscr{I}}(X)$ and $C_0^{\mathscr{I}}(X)$ further and deeper.

5. Continuous mappings with measure-zero cozero-set

This section comprises our first set of examples; it deals with continuous mappings with measure-zero cozero-set. The natural setting to state and prove our results is the one of topological measure spaces.

Notation 5.1. Let $(X, \mathscr{B}, \mu)$ be a measure space. Denote

$$\mathscr{M} = \big\{B \in \mathscr{B} : \mu(B) = 0\big\}.$$

Note that if $(X, \mathscr{B}, \mu)$ is a measure space then $(\mathscr{B}, \subseteq)$ is an upper semi-lattice (indeed, $\bigvee \mathscr{C} = \bigcup \mathscr{C}$ for any countable subset $\mathscr{C}$ of $\mathscr{B}$). The following holds trivially.

Lemma 5.2. *Let $(X, \mathscr{B}, \mu)$ be a measure space. Then $\mathscr{M}$ is a σ-ideal in $(\mathscr{B}, \subseteq)$.*

A *topological measure space* is a quadruple $(X, \mathscr{O}, \mathscr{B}, \mu)$ where $(X, \mathscr{B}, \mu)$ is a measure space and $(X, \mathscr{O})$ is a topological space such that $\mathscr{O} \subseteq \mathscr{B}$, that is, every open set (and thus every Borel set) is measurable. If $(X, \mathscr{O}, \mathscr{B}, \mu)$ is a topological measure space, by X (if used without any specification) we mean the topological space $(X, \mathscr{O})$.

Definition 5.3. Let $(X, \mathscr{O}, \mathscr{B}, \mu)$ be a topological measure space. The measure μ is said to be *locally null* if every $x \in X$ has a μ-null neighborhood in X.

The following follows trivially from the definitions.

Lemma 5.4. *Let $(X, \mathscr{O}, \mathscr{B}, \mu)$ be a topological measure space. Then X is locally null (with respect to the ideal $\mathscr{M}$) if and only if μ is locally null.*

The following two theorems are the main results of this section.

Theorem 5.5. *Let $(X, \mathscr{O}, \mathscr{B}, \mu)$ be a topological measure space and let*

$$\mathfrak{M}_{00}(X) = \big\{f \in C_b(X) : \operatorname{supp}(f) \textit{ has a } \mu\textit{-null neighborhood in } X\big\}.$$

Then

- *$\mathfrak{M}_{00}(X)$ is an ideal (in particular, a normed subalgebra) in the algebra $C_b(X)$.*

If X is completely regular, then

- *$\mathfrak{M}_{00}(X)$ is of empty hull if and only if μ is locally null.*

If X is completely regular and μ is locally null, then

- *$\mathfrak{M}_{00}(X)$ is unital if and only if μ is trivial.*

If X is normal and μ is locally null, then

- *The normed algebra $\mathfrak{M}_{00}(X)$ is isometrically isomorphic to $C_{00}(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y, namely*

$$Y=\bigcup\big\{\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C:C\in\mathrm{Coz}(X)\text{ and }\mathrm{cl}_X C\text{ has a }\mu\text{-null neighborhood in }X\big\},$$

  considered as a subspace of βX. Furthermore,
  - *X is dense in Y.*
  - *$\mathfrak{M}_{00}(X)$ is unital if and only if Y is compact.*

Proof. The theorem follows from Lemmas 5.2 and 5.4 and Theorems 3.3, 3.8, 3.10 and 3.13. □

Theorem 5.6. *Let $(X,\mathscr{O},\mathscr{B},\mu)$ be a topological measure space and let*

$$\mathfrak{M}_0(X)=\big\{f\in C_b(X):\mu\big(\mathrm{Coz}(f)\big)=0\big\}.$$

Then

- *$\mathfrak{M}_0(X)$ is a closed ideal (in particular, a Banach subalgebra) in the normed algebra $C_b(X)$.*
- *$\mathfrak{M}_{00}(X)\subseteq\mathfrak{M}_0(X)$.*

If X is completely regular, then

- *$\mathfrak{M}_0(X)$ is of empty hull if and only if μ is locally null.*

If X is completely regular and μ is locally null, then

- *$\mathfrak{M}_0(X)$ is unital if and only if μ is locally null.*

If X is normal and μ is locally null, then

- *The Banach algebra $\mathfrak{M}_0(X)$ is isometrically isomorphic to $C_0(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y, namely*

$$Y=\bigcup\big\{\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C:C\in\mathrm{Coz}(X)\text{ and }\mathrm{cl}_X C\text{ has a }\mu\text{-null neighborhood in }X\big\},$$

  considered as a subspace of βX. Furthermore,
  - *X is dense in Y.*
  - *$\mathfrak{M}_{00}(X)$ is dense in $\mathfrak{M}_0(X)$.*
  - *$\mathfrak{M}_0(X)$ is unital if and only if Y is compact.*

Proof. Note that $\mathrm{Coz}(X)\subseteq\mathscr{B}$. Thus $\mathfrak{M}_0(X)=C_0^{\mathscr{M}}(X)$ by Lemma 5.2 and Proposition 4.4. The remaining parts of the theorem follow from Lemma 5.4 and Theorems 4.5, 4.6, 4.7 and 4.9. □

Remark 5.7. Let $(X,\mathscr{O},\mathscr{B},\mu)$ be a topological measure space. Let

$$\mathscr{H}=\big\{B\in\mathscr{B}:\mu\big(\mathrm{cl}_X(B)\big)=0\big\}.$$

Then $\mathscr{H}$ is an ideal in $(\mathscr{B},\subseteq)$. One can now state and prove results analogous to Theorems 5.5 and 5.6 with the ideal $\mathscr{H}$ in place of the σ-ideal $\mathscr{M}$.

6. Closed z-ideals in $C_b(X)$ with empty hull

Let X be a regular Lindelöf space and let H be a closed z-ideal in $C_b(X)$ with empty hull. (Recall that an ideal H of $C_b(X)$ is a z-ideal if $\mathrm{Z}(f)=\mathrm{Z}(h)$, where $f\in C_b(X)$ and $h\in H$, implies that $f\in H$.) Let $\mathscr{L}=\mathrm{Coz}(X)$. Note that $(\mathscr{L},\subseteq)$ is a lattice closed under countable unions. (See Section 2.) Let

$$\mathscr{H}=\big\{\mathrm{Coz}(h):h\in H\big\}.$$

We show that $\mathscr{H}$ is a σ-ideal in $\mathscr{L}$. Suppose that $\mathrm{Coz}(f)\subseteq\mathrm{Coz}(h)$ with $f\in C_b(X)$ and $h\in H$. Then $\mathrm{Coz}(f)=\mathrm{Coz}(fh)$. In particular, $\mathrm{Z}(f)=\mathrm{Z}(fh)$. Since $fh\in H$ and H is a z-ideal, it follows that $f\in H$. Next, suppose that $f_1,f_2,\ldots\in H$ and let

$$f=\sum_{n=1}^{\infty}\frac{f_n^2}{2^n\|f_n^2\|}.$$

(We may assume that $f_n\neq\mathbf{0}$ for each positive integer n.) Then $f\in H$, as f is the limit of a sequence in H and by our assumption H is closed. Observe that

$$\mathrm{Coz}(f)=\bigcup_{n=1}^{\infty}\mathrm{Coz}(f_n).$$

Next, since H is of empty hull, for each $x\in X$ we have $x\notin\mathrm{Z}(h)$ for some $h\in H$, and thus $x\in\mathrm{Coz}(h)\in\mathscr{H}$. Therefore X is locally null (with respect to $\mathscr{H}$). By definition, we have

$$C_{00}^{\mathscr{H}}(X)=\big\{f\in C_b(X):\mathrm{supp}(f)\subseteq\mathrm{Coz}(h)\text{ for some }h\in H\big\}.$$

Note that if $f\in C_{00}^{\mathscr{H}}(X)$ then $\mathrm{Coz}(f)\subseteq\mathrm{Coz}(h)$ for some $h\in H$, and then arguing as in the above we have $f\in H$. That is $C_{00}^{\mathscr{H}}(X)\subseteq H$. Also, note that $C_{00}^{\mathscr{H}}(X)=C_{0}^{\mathscr{H}}(X)$ by Theorem 4.11, with $C_{00}^{\mathscr{H}}(X)$ and $C_{0}^{\mathscr{H}}(X)$, respectively, isometrically isomorphic to $C_{00}(Y)$ and $C_0(Y)$, and $C_{00}(Y)=C_0(Y)$, where Y is a locally compact countably compact Hausdorff space. By Theorem 4.7, either $C_{00}^{\mathscr{H}}(X)$ or $C_{0}^{\mathscr{H}}(X)$ is unital if and only if $\mathscr{H}$ is non-proper if and only if $X\in\mathscr{H}$ if and only if $X=\mathrm{Coz}(h)$ for some $h\in H$. We summarize our discussion in the following.

Theorem 6.1. *Let X be a regular Lindelöf space. Then every closed z-ideal H in $C_b(X)$ with empty hull contains a Banach subalgebra K of the form $C_0(Y)=C_{00}(Y)$ where Y is a locally compact countably compact Hausdorff space. Furthermore, K is unital if and only if H contains an element not vanishing on X.*

7. Subalgebras of ℓ_∞

In this section we consider certain ideals in $(\mathscr{P}(\mathbb{N}),\subseteq)$ (with $\mathbb{N}$ endowed with the discrete topology). This leads to introduction of certain subalgebras of ℓ_∞.

By ℓ_∞, c_0 and c_{00}, respectively, we denote the set of all bounded sequences in $\mathbb{R}$, the set of all vanishing sequences in $\mathbb{R}$, and the set of all sequences in $\mathbb{R}$ with only finitely many non-zero terms. Note that $\ell_\infty=C_b(\mathbb{N})$, $c_0=C_0(\mathbb{N})$ and $c_{00}=C_{00}(\mathbb{N})$, if $\mathbb{N}$ is given the discrete topology.

Let

$$\mathscr{S}=\bigg\{A\subseteq\mathbb{N}:\sum_{n\in A}\frac{1}{n}\text{ converges}\bigg\}.$$

Then $\mathscr{S}$ is an ideal in $(\mathscr{P}(\mathbb{N}),\subseteq)$, called the *summable ideal* in $\mathbb{N}$. A subset of $\mathbb{N}$ is called *small* if it is null (with respect to $\mathscr{S}$).

Note that there exists a family $\{A_i : i < 2^\omega\}$ consisting of infinite subsets of $\mathbb{N}$ such that the intersection $A_i \cap A_j$ is finite for any distinct $i, j < 2^\omega$. To see this, arrange the rational numbers into a sequence $q_1, q_2, \dots$ and for each $i \in \mathbb{R}$ define $A_i = \{n_1, n_2, \dots\}$ where $q_{n_1}, q_{n_2}, \dots$ is a subsequence of $q_1, q_2, \dots$ which converges to i. This known fact will be used in the proof of the following.

Recall that for a collection $\{X_i : i \in I\}$ of algebras the direct sum $\bigoplus_{i\in I} X_i$ is the set of all sequences $\{x_i\}_{i\in I}$ where $x_i \in X_i$ for each $i \in I$ such that $x_i = 0$ for all but a finite number of indices $i \in I$. The set $\bigoplus_{i\in I} X_i$ is an algebra with addition, multiplication and scalar multiplication defined component-wise. We denote the sequence $\{x_i\}_{i\in I}$ by a sum $\sum_{i\in I} x_i$. The direct sum $\bigoplus_{i\in I} X_i$ of a collection $\{X_i : i \in I\}$ of normed spaces is defined analogously and is a normed space with the norm given by

$$\Big\| \sum_{i\in I} x_i \Big\| = \sup\big\{\|x_i\|_{X_i} : i \in I\big\}.$$

Theorem 7.1. *Let*

$$\mathfrak{s}_{00} = \bigg\{\mathbf{x} \in \ell_\infty : \sum_{\mathbf{x}(n)\neq 0} \frac{1}{n} \textit{ converges}\bigg\}.$$

Then

(1) $\mathfrak{s}_{00}$ *is an ideal (in particular, a normed subalgebra) in* ℓ_∞.
(2) $\mathfrak{s}_{00}$ *is non-unital.*
(3) $\mathfrak{s}_{00}$ *is isometrically isomorphic to* $C_{00}(Y)$ *where*

$$Y = \bigcup \bigg\{\mathrm{cl}_{\beta\mathbb{N}} A : A \subseteq \mathbb{N} \textit{ and } \sum_{n\in A} \frac{1}{n} \textit{ converges}\bigg\},$$

considered as a subspace of $\beta\mathbb{N}$.
(4) $\mathfrak{s}_{00}$ *contains a copy of the normed algebra* $\bigoplus_{n=1}^\infty \ell_\infty$.
(5) $\mathfrak{s}_{00}/c_{00}$ *contains a copy of the algebra*

$$\bigoplus_{i<2^\omega} \frac{\ell_\infty}{c_{00}}.$$

Proof. (1)–(3). Consider the ideal $\mathscr{S}$ of $(\mathscr{P}(\mathbb{N}), \subseteq)$. Note that if $\mathbf{x} \in \ell_\infty$ (since $\mathbb{N}$ is discrete) then

$$\mathrm{supp}(\mathbf{x}) = \big\{n \in \mathbb{N} : \mathbf{x}(n) \neq 0\big\}$$

and $\mathrm{supp}(\mathbf{x})$ is null if and only if it has a null neighborhood in $\mathbb{N}$. Thus $\mathfrak{s}_{00} = C_{00}^{\mathscr{S}}(\mathbb{N})$. It now follows from Theorem 3.3 that $\mathfrak{s}_{00}$ is an ideal in ℓ_∞. The remaining parts follow from Theorem 3.13. Observe that $\mathbb{N}$ is locally null (indeed, $\{n\}$ is a null neighborhood of n in $\mathbb{N}$ for each $n \in \mathbb{N}$) and that $\mathscr{S}$ is non-proper (as $\sum 1/n$ diverges). Also, note that (since $\mathbb{N}$ is discrete) every subset A of $\mathbb{N}$ is a cozero-set in $\mathbb{N}$ and since A is open-closed in $\mathbb{N}$ its closure $\mathrm{cl}_{\beta\mathbb{N}} A$ in $\beta\mathbb{N}$ is open in $\beta\mathbb{N}$. Therefore $Y = \lambda_{\mathscr{S}}\mathbb{N}$.

(4). By (3), we may consider $C_{00}(Y)$ in place of $\mathfrak{s}_{00}$. Let A be an infinite subset of $\mathbb{N}$ such that $\sum_{n\in A} 1/n$ converges (which exists, for example, let $A = \{2^n : n \in \mathbb{N}\}$). Let $A_1, A_2, \dots$ be a partition of A into pairwise disjoint infinite subsets. Let $n = 1, 2, \dots$. We may assume that $C(\mathrm{cl}_{\beta\mathbb{N}} A_n)$ is a subalgebra of $C_{00}(Y)$. (Since A_n is open-closed in $\mathbb{N}$ it has open-closed closure $\mathrm{cl}_{\beta\mathbb{N}} A_n$ in $\beta\mathbb{N}$. Thus each element of $C(\mathrm{cl}_{\beta\mathbb{N}} A_n)$ may be continuously extended over Y by defining it to be identically 0

elsewhere.) Note that $\mathrm{cl}_{\beta\mathbb{N}}A_i$ and $\mathrm{cl}_{\beta\mathbb{N}}A_j$ are disjoint for any distinct $i,j=1,2,\ldots$, as A_i and A_j are disjoint open-closed subspaces (and thus zero-sets) of $\mathbb{N}$. Thus, the inclusion mapping

$$\iota:\bigoplus_{n=1}^{\infty}C(\mathrm{cl}_{\beta\mathbb{N}}A_n)\rightarrow C_{00}(Y)$$

is an algebra isomorphism (onto its image) and it preserves norms. (For the latter, use the fact that $\mathrm{cl}_{\beta\mathbb{N}}A_i$'s are disjoint for distinct indices.) Note that $\mathrm{cl}_{\beta\mathbb{N}}A_n$ coincides with βA_n, as A_n is closed in the normal space $\mathbb{N}$. Finally, observe that

$$C(\mathrm{cl}_{\beta\mathbb{N}}A_n)=C(\beta A_n)=C(\beta\mathbb{N})=C_b(\mathbb{N})=\ell_\infty.$$

(5). By (3), we may consider $C_{00}(Y)$ in place of $\mathfrak{s}_{00}$. Let A be an infinite subset of $\mathbb{N}$ such that $\sum_{n\in A}1/n$ converges. Consider a family $\{A_i:i<2^\omega\}$ consisting of infinite subsets of A such that $A_i\cap A_j$ is finite for any distinct $i,j<2^\omega$. Let

$$H=\big\{f\in C_{00}(Y):\mathrm{supp}(f)\subseteq A\big\}$$

and let

$$H_i=\big\{f\in C(\mathrm{cl}_{\beta\mathbb{N}}A_i):\mathrm{supp}(f)\subseteq A_i\big\}$$

for each $i<2^\omega$. As in (4), we may assume that $C(\mathrm{cl}_{\beta\mathbb{N}}A_i)$ is a subalgebra of $C_{00}(Y)$ for each $i<2^\omega$, and thus, we may assume that $H_i\subseteq H$. Note that if $f\in H$ then $\mathrm{supp}(f)$ is finite, as it is a compact subspace of $\mathbb{N}$.

Define

$$\Theta:\bigoplus_{i<2^\omega}\frac{C(\mathrm{cl}_{\beta\mathbb{N}}A_i)}{H_i}\rightarrow\frac{C_{00}(Y)}{H}$$

by

$$\sum_{i<2^\omega}(f_i+H_i)\mapsto\sum_{i<2^\omega}f_i+H$$

where $f_i\in C(\mathrm{cl}_{\beta\mathbb{N}}A_i)$ for each $i<2^\omega$. We show that Θ is an isometric isomorphism onto its image; since

$$\frac{C(\mathrm{cl}_{\beta\mathbb{N}}A_i)}{H_i}=\frac{\ell_\infty}{c_{00}}$$

for each $i<2^\omega$, this completes the proof.

First, note that Θ is well defined; to show this, let

$$\sum_{i<2^\omega}(f_i+H_i)=\sum_{i<2^\omega}(g_i+H_i)$$

where $f_i,g_i\in C(\mathrm{cl}_{\beta\mathbb{N}}A_i)$ for each $i<2^\omega$. For each $i<2^\omega$ then $f_i+H_i=g_i+H_i$, or equivalently $f_i-g_i\in H_i$, in particular $f_i-g_i\in H$. Thus

$$\sum_{i<2^\omega}(f_i-g_i)\in H$$

and therefore

$$\Theta\bigg(\sum_{i<2^\omega}(f_i+H_i)\bigg)=\sum_{i<2^\omega}f_i+H=\sum_{i<2^\omega}g_i+H=\Theta\bigg(\sum_{i<2^\omega}(g_i+H_i)\bigg).$$

Now, we show that Θ preserves product. Let $f_i,g_i\in C(\mathrm{cl}_{\beta\mathbb{N}}A_i)$ for each $i<2^\omega$. Then

$$\Theta\bigg(\sum_{i<2^\omega}(f_i+H_i)\cdot\sum_{i<2^\omega}(g_i+H_i)\bigg)=\Theta\bigg(\sum_{i<2^\omega}(f_ig_i+H_i)\bigg)=\sum_{i<2^\omega}f_ig_i+H.$$

Note that if $k,l<2^\omega$ with $k\neq l$ then

$$\begin{aligned}\operatorname{supp}(f_kg_l) &\subseteq \operatorname{supp}(f_k)\cap\operatorname{supp}(g_l)\\ &\subseteq \operatorname{cl}_{\beta\mathbb{N}}A_k\cap\operatorname{cl}_{\beta\mathbb{N}}A_l=\operatorname{cl}_{\beta\mathbb{N}}(A_k\cap A_l)=A_k\cap A_l\subseteq A\end{aligned}$$

and thus $f_kg_l\in H$. We have

$$\begin{aligned}\Theta\bigg(\sum_{i<2^\omega}(f_i+H_i)\bigg)\cdot\Theta\bigg(\sum_{i<2^\omega}(g_i+H_i)\bigg) &= \bigg(\sum_{i<2^\omega}f_i+H\bigg)\cdot\bigg(\sum_{i<2^\omega}g_i+H\bigg)\\ &= \bigg(\sum_{i<2^\omega}f_i\sum_{i<2^\omega}g_i\bigg)+H\\ &= \bigg(\sum_{i<2^\omega}f_ig_i+\sum_{k\neq l}f_kg_l\bigg)+H\\ &= \sum_{i<2^\omega}f_ig_i+H.\end{aligned}$$

This together with the above proves that

$$\Theta\bigg(\sum_{i<2^\omega}(f_i+H_i)\cdot\sum_{i<2^\omega}(g_i+H_i)\bigg)=\Theta\bigg(\sum_{i<2^\omega}(f_i+H_i)\bigg)\cdot\Theta\bigg(\sum_{i<2^\omega}(g_i+H_i)\bigg).$$

That Θ preserves addition and scalar multiplication follows analogously.

Next, we show that Θ is injective. Let

$$\Theta\bigg(\sum_{i<2^\omega}(f_i+H_i)\bigg)=0$$

where $f_i\in C(\operatorname{cl}_{\beta\mathbb{N}}A_i)$ for each $i<2^\omega$. Then

$$\sum_{i<2^\omega}f_i+H=0,$$

or, equivalently

$$h=\sum_{i<2^\omega}f_i\in H.$$

Suppose that $f_{i_1},\ldots,f_{i_n}$ are the possibly non-zero terms. Fix some $k=1,\ldots,n$. Then

$$f_{i_k}=h-\sum_{1\leq j\neq k\leq n}f_{i_j}.$$

We have

$$\operatorname{Coz}(f_{i_k})\subseteq\operatorname{Coz}(h)\cup\bigcup_{1\leq j\neq k\leq n}\operatorname{Coz}(f_{i_j})\subseteq\operatorname{supp}(h)\cup\bigcup_{1\leq j\neq k\leq n}\operatorname{cl}_{\beta\mathbb{N}}A_{i_j}.$$

Note that the latter set is compact, as $\operatorname{supp}(h)$ is finite, since $h\in H$. Therefore

$$\operatorname{supp}(f_{i_k})\subseteq\operatorname{supp}(h)\cup\bigcup_{1\leq j\neq k\leq n}\operatorname{cl}_{\beta\mathbb{N}}A_{i_j}. \tag{7.1}$$

Intersecting both sides of (7.1) with $\operatorname{cl}_{\beta\mathbb{N}}A_{i_k}$ yields

$$\begin{aligned}\operatorname{supp}(f_{i_k}) &\subseteq \operatorname{supp}(h)\cup\bigcup_{1\leq j\neq k\leq n}(\operatorname{cl}_{\beta\mathbb{N}}A_{i_j}\cap\operatorname{cl}_{\beta\mathbb{N}}A_{i_k})\\ &= \operatorname{supp}(h)\cup\bigcup_{1\leq j\neq k\leq n}\operatorname{cl}_{\beta\mathbb{N}}(A_{i_j}\cap A_{i_k})=\operatorname{supp}(h)\cup\bigcup_{1\leq j\neq k\leq n}(A_{i_j}\cap A_{i_k})\end{aligned}$$

with the latter being a subset of $\mathbb{N}$. Thus

$$\operatorname{supp}(f_{i_k}) \subseteq \mathbb{N} \cap \mathrm{cl}_{\beta\mathbb{N}} A_{i_k} = A_{i_k}$$

and therefore $f_{i_k} \in H_{i_k}$. This implies that

$$\sum_{j=1}^{n} (f_{i_j} + H_{i_j}) = 0.$$

Thus Θ is injective. □

Theorem 7.2. *Let*

$$\mathfrak{s}_0 = \bigg\{ \mathbf{x} \in \ell_\infty : \sum_{|\mathbf{x}(n)| \geq \epsilon} \frac{1}{n} \text{ converges for each } \epsilon > 0 \bigg\}.$$

Then

(1) $\mathfrak{s}_0$ *is a closed ideal (in particular, a Banach subalgebra) in* ℓ_∞.
(2) $\mathfrak{s}_0$ *is non-unital.*
(3) $\mathfrak{s}_0$ *contains* $\mathfrak{s}_{00}$ *as a dense subspace.*
(4) $\mathfrak{s}_0$ *is isometrically isomorphic to* $C_0(Y)$ *where*

$$Y = \bigcup \bigg\{ \mathrm{cl}_{\beta\mathbb{N}} A : A \subseteq \mathbb{N} \text{ and } \sum_{n \in A} \frac{1}{n} \text{ converges} \bigg\},$$

considered as a subspace of $\beta\mathbb{N}$.
(5) $\mathfrak{s}_0 / c_0$ *contains a copy of the normed algebra*

$$\bigoplus_{i < 2^\omega} \frac{\ell_\infty}{c_0}.$$

Proof. The proofs for (1)–(4) are analogous to the proofs for the corresponding parts in Theorem 7.1, making use of Theorems 4.5 and 4.9. Note that if $\mathbf{x} \in \ell_\infty$ and $\epsilon > 0$ then $|\mathbf{x}|^{-1}([\epsilon, \infty))$ is null (with respect to $\mathscr{S}$) if and only if

$$\sum_{|\mathbf{x}(n)| \geq \epsilon} \frac{1}{n}$$

converges. Thus, in particular $\mathfrak{s}_0 = C_0^{\mathscr{S}}(\mathbb{N})$.

(5). The proof of this part is analogous to the proof of the corresponding part in Theorem 7.1; we will highlight only the differences.

Let A and $\{A_i : i < 2^\omega\}$ be as chosen in the proof of Theorem 7.1. Let

$$H = \big\{ f \in C_0(Y) : |f|^{-1}\big([\epsilon, \infty)\big) \subseteq A \text{ for each } \epsilon > 0 \big\}$$

and let

$$H_i = \big\{ f \in C(\mathrm{cl}_{\beta\mathbb{N}} A_i) : |f|^{-1}\big([\epsilon, \infty)\big) \subseteq A_i \text{ for each } \epsilon > 0 \big\}$$

for each $i < 2^\omega$. We consider $C(\mathrm{cl}_{\beta\mathbb{N}} A_i)$ as a subalgebra of $C_0(Y)$, and thus, we may assume that $H_i \subseteq H$ for each $i < 2^\omega$. Note that if $i < 2^\omega$ and $f \in H_i$ then $|f|^{-1}([\epsilon, \infty))$ is a finite subset of A_i, as it is compact (since it is closed in $\mathrm{cl}_{\beta\mathbb{N}} A_i$). Define

$$\Theta : \bigoplus_{i < 2^\omega} \frac{C(\mathrm{cl}_{\beta\mathbb{N}} A_i)}{H_i} \to \frac{C_0(Y)}{H}$$

by

$$\sum_{i < 2^\omega} (f_i + H_i) \mapsto \sum_{i < 2^\omega} f_i + H$$

where $f_i \in C(\mathrm{cl}_{\beta\mathbb{N}} A_i)$ for each $i < 2^\omega$. Arguing as in the proof of Theorem 7.1 it follows that Θ is well defined. The proofs that Θ preserves product is analogous to the corresponding part in the proof of Theorem 7.1; simply observe that if $f_i, g_i \in C(\mathrm{cl}_{\beta\mathbb{N}} A_i)$ for each $i < 2^\omega$ and $\epsilon > 0$, then for any $k, l < 2^\omega$ with $k \neq l$ we have

$$\begin{aligned} |f_k g_l|^{-1}\big([\epsilon,\infty)\big) &\subseteq \mathrm{Coz}(f_k g_l) \\ &\subseteq \mathrm{Coz}(f_k) \cap \mathrm{Coz}(g_l) \\ &\subseteq \mathrm{cl}_{\beta\mathbb{N}} A_k \cap \mathrm{cl}_{\beta\mathbb{N}} A_l = \mathrm{cl}_{\beta\mathbb{N}}(A_k \cap A_l) = A_k \cap A_l \subseteq A \end{aligned}$$

and thus $f_k g_l \in H$. The proofs that Θ preserves addition and scalar multiplication are analogous.

Now, we show that Θ is injective. Let

$$\Theta\bigg(\sum_{i<2^\omega}(f_i + H_i)\bigg) = 0$$

where $f_i \in C(\mathrm{cl}_{\beta\mathbb{N}} A_i)$ for each $i < 2^\omega$. Suppose that $f_{i_1}, \ldots, f_{i_n}$ are the possibly non-zero terms. Fix some $k = 1, \ldots, n$. Then, as in the proof of Theorem 7.1 we have

$$f_{i_k} = h - \sum_{1 \leq j \neq k \leq n} f_{i_j}$$

for some $h \in H$. If $\epsilon > 0$ then

$$\begin{aligned} |f_{i_k}|^{-1}\big([\epsilon,\infty)\big) &\subseteq |h|^{-1}\big([\epsilon/n,\infty)\big) \cup \bigcup_{1 \leq j \neq k \leq n} |f_{i_j}|^{-1}\big([\epsilon/n,\infty)\big) \\ &\subseteq A \cup \bigcup_{1 \leq j \neq k \leq n} \mathrm{cl}_{\beta\mathbb{N}} A_{i_j}. \end{aligned}$$

Arguing as in the proof of Theorem 7.1, intersecting both sides of the above with $\mathrm{cl}_{\beta\mathbb{N}} A_{i_k}$ yields

$$|f_{i_k}|^{-1}\big([\epsilon,\infty)\big) \subseteq A_{i_k}$$

and therefore $f_{i_k} \in H_{i_k}$. This implies that

$$\sum_{j=1}^{n}(f_{i_j} + H_{i_j}) = 0.$$

Thus Θ is injective.

Next, we show that Θ is an isometry. First, we need to show the following.

Claim. *Let $i < 2^\omega$ and $f \in C(\mathrm{cl}_{\beta\mathbb{N}} A_i)$. Then*

$$\|f + H_i\| = \big\|f|_{Y\setminus\mathbb{N}}\big\|_\infty.$$

Proof of the claim. Suppose in the contrary that

$$\|f + H_i\| < \big\|f|_{Y\setminus\mathbb{N}}\big\|_\infty. \tag{7.2}$$

Then

$$\alpha = \|f + h\|_\infty < \big\|f|_{Y\setminus\mathbb{N}}\big\|_\infty$$

for some $h \in H_i$. Let $\alpha < \gamma < \|f|_{Y\setminus\mathbb{N}}\|_\infty$. Then

$$B = |f|^{-1}\big([\gamma,\infty)\big) \subseteq |h|^{-1}\big([\gamma - \alpha,\infty)\big) = C;$$

as if $y\in Y$ such that $|f(y)|\geq\gamma$, then
$$\alpha\geq\big|f(y)+h(y)\big|\geq\big|f(y)\big|-\big|h(y)\big|\geq\gamma-\big|h(y)\big|$$
and thus $|h(y)|\geq\gamma-\alpha$. Note that B is a finite subset of $\mathbb{N}$, as C is so, since $h\in H_i$. We have
$$Y=\mathrm{cl}_Y\mathbb{N}=B\cup\mathrm{cl}_Y(\mathbb{N}\backslash B)$$
and thus
$$\begin{aligned}|f|(Y\backslash\mathbb{N})\subseteq|f|\big(\mathrm{cl}_Y(\mathbb{N}\backslash B)\big)\subseteq\overline{|f|(\mathbb{N}\backslash B)} &= \overline{|f|\big(\mathbb{N}\cap|f|^{-1}\big([0,\gamma)\big)\big)}\\ &\subseteq \overline{|f|\big(|f|^{-1}\big([0,\gamma)\big)\big)}\subseteq\overline{[0,\gamma)}=[0,\gamma],\end{aligned}$$
where the bar denotes the closure in $\mathbb{R}$. This implies that $\|f|_{Y\backslash\mathbb{N}}\|_\infty\leq\gamma$, which contradicts the choice of γ. Thus (7.2) is false, that is
$$\|f+H_i\|\geq\big\|f|_{Y\backslash\mathbb{N}}\big\|_\infty. \tag{7.3}$$

Now, we prove the reverse inequality in (7.3). Let $\delta=\|f|_{Y\backslash\mathbb{N}}\|_\infty$. We first show that
$$D_\epsilon=\mathbb{N}\cap|f|^{-1}\big((\delta+\epsilon,\infty)\big)$$
is finite for every $\epsilon>0$. Suppose the contrary, that is, suppose that D_ϵ is infinite for some $\epsilon>0$. Note that $\mathrm{cl}_{\beta\mathbb{N}}D_\epsilon\backslash\mathbb{N}$ is non-empty, as $\mathrm{cl}_{\beta\mathbb{N}}D_\epsilon\subseteq\mathbb{N}$ implies that $D_\epsilon=\mathrm{cl}_{\beta\mathbb{N}}D_\epsilon$ is compact and is then finite. Let $p\in\mathrm{cl}_{\beta\mathbb{N}}D_\epsilon\backslash\mathbb{N}$. Note that
$$\mathrm{cl}_{\beta\mathbb{N}}D_\epsilon=\mathrm{cl}_{\beta\mathbb{N}}\big(|f|^{-1}\big((\delta+\epsilon,\infty)\big)\big).$$
We have
$$\begin{aligned}|f|(p)\in|f|(\mathrm{cl}_{\beta\mathbb{N}}D_\epsilon) &= |f|\big(\mathrm{cl}_{\beta\mathbb{N}}\big(|f|^{-1}\big((\delta+\epsilon,\infty)\big)\big)\big)\\ &\subseteq \overline{|f|\big(|f|^{-1}\big((\delta+\epsilon,\infty)\big)\big)}\subseteq\overline{(\delta+\epsilon,\infty)}=[\delta+\epsilon,\infty),\end{aligned}$$
where the bar denotes the closure in $\mathbb{R}$. Therefore
$$\delta=\big\|f|_{Y\backslash\mathbb{N}}\big\|_\infty\geq|f(p)|\geq\delta+\epsilon,$$
which is not possible. This shows that D_ϵ is finite for every $\epsilon>0$. Now, let $\epsilon>0$. Define $h:Y\rightarrow\mathbb{R}$ such that $h(x)=-f(x)$ if $x\in D_\epsilon$ and $h(x)=0$ otherwise. Note that D_ϵ is closed in Y, as it is finite, and D_ϵ is open in Y, as it is open in $\mathbb{N}$ and $\mathbb{N}$ is open in $\beta\mathbb{N}$ (and thus in Y), since $\mathbb{N}$ is locally compact. Therefore h is continuous. Observe that
$$\mathrm{Coz}(h)\subseteq D_\epsilon\subseteq\mathbb{N}\cap|f|^{-1}\big([\delta+\epsilon,\infty)\big)\subseteq\mathbb{N}\cap\mathrm{cl}_{\beta\mathbb{N}}A_i=A_i.$$
Thus $h\in H_i$. Note that $f+h\equiv\mathbf{0}$ on D_ϵ and $f+h\equiv f$ on $Y\backslash D_\epsilon$. Also,
$$\big\|f|_{\mathbb{N}\backslash D_\epsilon}\big\|_\infty\leq\delta+\epsilon$$
by the way we have defined D_ϵ. Now
$$\|f+H_i\|\leq\|f+h\|_\infty=\big\|f|_{Y\backslash D_\epsilon}\big\|_\infty=\max\big\{\big\|f|_{\mathbb{N}\backslash D_\epsilon}\big\|_\infty,\big\|f|_{Y\backslash\mathbb{N}}\big\|_\infty\big\}\leq\delta+\epsilon.$$
Since $\epsilon>0$ is arbitrary, it follows that
$$\|f+H_i\|\leq\delta=\big\|f|_{Y\backslash\mathbb{N}}\big\|_\infty.$$
This together with (7.3) proves the claim.

Claim. *Let $i_1,\dots,i_n<2^\omega$ and $f_{i_j}\in C(\mathrm{cl}_{\beta\mathbb{N}}A_{i_j})$ for each $j=1,\dots,n$. Then*
$$\Big\|\sum_{j=1}^n f_{i_j}+H\Big\|=\Big\|\Big(\sum_{j=1}^n f_{i_j}\Big)\Big|_{Y\setminus\mathbb{N}}\Big\|_\infty.$$

Proof of the claim. This follows by an argument similar to the one we have given in the first claim; one simply needs to replace f by $\sum_{j=1}^n f_{i_j}$, H_i by H, and A_i by A throughout.

Now, let $i_1,\dots,i_n<2^\omega$ and $f_{i_j}\in C(\mathrm{cl}_{\beta\mathbb{N}}A_{i_j})$ for each $j=1,\dots,n$. For simplicity of the notation let
$$f=\sum_{j=1}^n f_{i_j}.$$
Let $1\le j\ne k\le n$. Then $A_j\cap A_k$ is finite and thus
$$\mathrm{Coz}(f_{i_k})\cap\mathrm{Coz}(f_{i_l})\subseteq\mathrm{cl}_{\beta\mathbb{N}}A_{i_k}\cap\mathrm{cl}_{\beta\mathbb{N}}A_{i_l}=\mathrm{cl}_{\beta\mathbb{N}}(A_{i_k}\cap A_{i_l})=A_{i_k}\cap A_{i_l}\subseteq\mathbb{N}.$$
In particular,
$$\mathrm{Coz}\big(f_{i_k}|_{Y\setminus\mathbb{N}}\big)\cap\mathrm{Coz}\big(f_{i_l}|_{Y\setminus\mathbb{N}}\big)=\emptyset.$$
Now, by the second claim
$$\Big\|\Theta\Big(\sum_{j=1}^n(f_{i_j}+H_{i_j})\Big)\Big\|=\|f+H\|=\big\|f|_{Y\setminus\mathbb{N}}\big\|_\infty$$
and by the first claim
$$\begin{aligned}\big\|f|_{Y\setminus\mathbb{N}}\big\|_\infty &= \max\big\{\big\|f|_{(\mathrm{cl}_{\beta\mathbb{N}}A_{i_j}\setminus\mathbb{N})}\big\|_\infty : j=1,\dots,n\big\}\\ &= \max\big\{\big\|f_{i_j}|_{Y\setminus\mathbb{N}}\big\|_\infty : j=1,\dots,n\big\}\\ &= \max\big\{\|f_{i_j}+H_{i_j}\| : j=1,\dots,n\big\}=\Big\|\sum_{j=1}^n(f_{i_j}+H_{i_j})\Big\|.\end{aligned}$$
That is, Θ is an isometry. □

Remark 7.3. Any sequence $f:\mathbb{N}\to(0,\infty)$ such that $\sum_{n=1}^\infty f(n)$ diverges, determines an ideal
$$\mathscr{S}_f=\Big\{A\subseteq\mathbb{N}:\sum_{n\in A}f(n)\text{ converges}\Big\}$$
in $(\mathscr{P}(\mathbb{N}),\subseteq)$. This provides a more general setting to state and prove Theorems 7.1 and 7.2.

Let
$$\mathscr{D}=\Big\{A\subseteq\mathbb{N}:\limsup_{n\to\infty}\frac{|A\cap\{1,\dots,n\}|}{n}=0\Big\}.$$
Then $\mathscr{D}$ also is an ideal in $(\mathscr{P}(\mathbb{N}),\subseteq)$, called the *density ideal* in $\mathbb{N}$. In other words, $\mathscr{D}$ consists of those subsets D of $\mathbb{N}$ such that D has *asymptotic density* zero. Note that every small set in $\mathbb{N}$ has asymptotic density zero, that is $\mathscr{S}\subseteq\mathscr{D}$; the converse, however, does not hold in general. The set of all prime numbers has asymptotic density zero but it is not small. (For more information on the subject, see [8].)

The following is dual to Theorems 7.1 and 7.2 and may be proved analogously, replacing the ideal $\mathscr{S}$ by the ideal $\mathscr{D}$ throughout the proofs already given.

Theorem 7.4. *Let*

$$\mathfrak{d}_{00}=\left\{\mathbf{x}\in\ell_\infty:\limsup_{n\to\infty}\frac{\big|\{k\leq n:\mathbf{x}(k)\neq 0\}\big|}{n}=0\right\}$$

and

$$\mathfrak{d}_{0}=\left\{\mathbf{x}\in\ell_\infty:\limsup_{n\to\infty}\frac{\big|\{k\leq n:|\mathbf{x}(k)|\geq \epsilon\}\big|}{n}=0 \text{ for each } \epsilon>0\right\}.$$

Then

(1) $\mathfrak{d}_{00}$ *is an ideal in* ℓ_∞ *and* $\mathfrak{d}_0$ *is a closed ideal in* ℓ_∞.
(2) $\mathfrak{d}_{00}$ *is dense in* $\mathfrak{d}_0$.
(3) *Neither* $\mathfrak{d}_{00}$ *nor* $\mathfrak{d}_0$ *is unital.*
(4) $\mathfrak{d}_{00}$ *and* $\mathfrak{d}_0$ *are isometrically isomorphic to* $C_{00}(Y)$ *and* $C_0(Y)$, *respectively, for the subspace*

$$Y=\bigcup\left\{\mathrm{cl}_{\beta\mathbb{N}}A:A\subseteq\mathbb{N} \text{ and } \limsup_{n\to\infty}\frac{\big|A\cap\{1,\ldots,n\}\big|}{n}=0\right\}$$

of $\beta\mathbb{N}$.
(5) $\mathfrak{d}_{00}$ *contains a copy of the normed algebra* $\bigoplus_{n=1}^\infty\ell_\infty$.
(6) $\mathfrak{d}_{00}/c_{00}$ *contains a copy of the algebra*

$$\bigoplus_{i<2^\omega}\frac{\ell_\infty}{c_{00}}.$$

(7) $\mathfrak{d}_0/c_0$ *contains a copy of the normed algebra*

$$\bigoplus_{i<2^\omega}\frac{\ell_\infty}{c_{0}}.$$

8. Continuous mappings whose support has a topological property $\mathfrak{P}$

Let X be a space and let $\mathfrak{P}$ be a topological property. Then

$$\mathscr{I}_\mathfrak{P}=\{A\subseteq X:\mathrm{cl}_XA \text{ has } \mathfrak{P}\}$$

is an ideal in $(\mathscr{P}(X),\subseteq)$ if $\mathfrak{P}$ is required to satisfy certain mild requirements. We consider $C_{00}^\mathscr{I}(X)$ and $C_0^\mathscr{I}(X)$ where $\mathscr{I}=\mathscr{I}_\mathfrak{P}$; for simplicity of the notation we denote them by $C_{00}^\mathfrak{P}(X)$ and $C_0^\mathfrak{P}(X)$, respectively, and denote $\lambda_\mathscr{I}X$ by $\lambda_\mathfrak{P}X$. In this context we will have

$$C_{00}^\mathfrak{P}(X)=\big\{f\in C_b(X):\mathrm{supp}(f) \text{ has a closed neighborhood in } X \text{ with } \mathfrak{P}\big\}$$

and

$$C_{0}^\mathfrak{P}(X)=\big\{f\in C_b(X):|f|^{-1}\big([1/n,\infty)\big) \text{ has } \mathfrak{P} \text{ for each } n\big\}.$$

The ideal $\mathscr{I}_\mathfrak{P}$ is non-proper if and only if X is non-$\mathfrak{P}$, and if X is regular, X is locally null if and only if X is locally-$\mathfrak{P}$. Particular attention is paid to spaces X and topological properties $\mathfrak{P}$ such that X is locally-$\mathfrak{P}$ and has $\mathfrak{Q}$ and $\mathfrak{P}$ satisfies

$$\mathfrak{P}+\mathfrak{Q}\rightarrow\text{The Lindelöf property},$$

where $\mathfrak{Q}$ is a topological property subject to some requirements. In particular, in this case we have

$$C_{00}^\mathfrak{P}(X)=\big\{f\in C_b(X):\mathrm{supp}(f) \text{ has } \mathfrak{P}\big\}.$$

The special case in which $\mathfrak{P}$ is the Lindelöf property and $\mathfrak{Q}$ is metrizability (or paracompactness) is studied in great detail. Among other things, we show that $\lambda_{\mathfrak{P}}X$ is countably compact and is non-normal if X is non-$\mathfrak{P}$. In particular,

$$\dim C_{00}^{\mathfrak{P}}(X) = \ell(X)^{\aleph_0},$$

where $\ell(X)$ is the Lindelöf number of X. The concluding results in this section deal with realcompactness and pseudocompactness. We show that if $\mathfrak{P}$ is realcompactness and X is normal, then

$$\lambda_{\mathfrak{P}}X = \beta X \backslash \mathrm{cl}_{\beta X}(\upsilon X \backslash X),$$

where υX is the Hewitt realcompactification of X. Also, if $\mathfrak{P}$ is pseudocompactness and X is completely regular, then

$$\lambda_{\mathscr{U}}X = \mathrm{int}_{\beta X}\upsilon X$$

for the ideal

$$\mathscr{U} = \big\{A \in \mathscr{RC}(X) : A \text{ is pseudocompact}\big\}$$

of $(\mathscr{RC}(X), \subseteq)$.

Now we proceed with the formal treatment of the subject. Theorem 8.8 improves results from [19] (also [20]), Theorem 8.12 is actually the main result of [16] rephrased in the new context, and Theorems 8.19 and 8.23 modify results from [13] and [18], respectively.

Definition 8.1. Let $\mathfrak{P}$ be a topological property. Then

- $\mathfrak{P}$ is *closed hereditary*, if any closed subspace of a space with $\mathfrak{P}$, also has $\mathfrak{P}$.
- $\mathfrak{P}$ is *preserved under finite* (resp. *countable*) *closed sums*, if any space which is expressible as a finite (resp. countable) union of its closed subspaces each having $\mathfrak{P}$, also has $\mathfrak{P}$.

Notation 8.2. Let X be a space and let $\mathfrak{P}$ be a topological property. Denote

$$\mathscr{I}_{\mathfrak{P}} = \{A \subseteq X : \mathrm{cl}_X A \text{ has } \mathfrak{P}\}.$$

We may use the following lemma without explicitly referring to it.

Lemma 8.3. *Let X be a space and let $\mathfrak{P}$ be a closed hereditary topological property preserved under finite closed sums. Then $\mathscr{I}_{\mathfrak{P}}$ is an ideal in $(\mathscr{P}(X), \subseteq)$.*

Proof. Let $B \subseteq A$ with $A \in \mathscr{I}_{\mathfrak{P}}$. Then $\mathrm{cl}_X B \subseteq \mathrm{cl}_X A$. Since $\mathrm{cl}_X A$ has $\mathfrak{P}$ and $\mathfrak{P}$ is closed hereditary, the closed subspace $\mathrm{cl}_X B$ of $\mathrm{cl}_X A$ also has $\mathfrak{P}$. That is $B \in \mathscr{I}_{\mathfrak{P}}$. Next, let $A_1, \ldots, A_n \in \mathscr{I}_{\mathfrak{P}}$. Then $\mathrm{cl}_X A_1 \cup \cdots \cup \mathrm{cl}_X A_n$ has $\mathfrak{P}$, as each $\mathrm{cl}_X A_1, \ldots, \mathrm{cl}_X A_n$ has $\mathfrak{P}$ and $\mathfrak{P}$ is preserved under finite closed sums. Since

$$\mathrm{cl}_X(A_1 \cup \cdots \cup A_n) = \mathrm{cl}_X A_1 \cup \cdots \cup \mathrm{cl}_X A_n$$

we then have $A_1 \cup \cdots \cup A_n \in \mathscr{I}_{\mathfrak{P}}$. ☐

Example 8.4. The list of topological properties $\mathfrak{P}$ which are closed hereditary and preserved under finite closed sums (thus satisfying the assumption of Lemma 8.3) is quite long and include almost all important covering properties (that is, topological properties described in terms of the existence of certain kinds of open subcovers or refinements of a given open cover of a certain type); among them are: (1) compactness (2) $[\theta, \kappa]$-compactness (in particular, countable compactness and the Lindelöf property) (3) paracompactness (4) metacompactness (5) countable

paracompactness (6) subparacompactness (7) θ-refinability (or submetacompactness) (8) the σ-para-Lindelöf property (9) $\delta\theta$-refinability (or the submeta-Lindelöf property) (10) weak θ-refinability, and finally (11) weak $\delta\theta$-refinability. (See [5], [25] and [27] for definitions.) These topological properties are all closed hereditary (see Theorem 7.1 of [5] for (3)–(4) and (6)–(11), Theorem 3.1 of [25] for (2) and Exercise 5.2.B of [7] for (5); this is obvious for (1)) and are preserved under finite closed sums (see Theorems 7.3 and 7.4 of [5] for (3)–(4) and (6)–(11), and Theorem 3.7.22 and Exercises 5.2.B and 5.2.G of [7] for (5); this follows from definition for (2) and is obvious for (1)).

There are examples of topological properties, not generally considered as a covering property, which are closed hereditary and preserved under finite closed sums. We only mention α-boundedness. (A space X is called *α-bounded*, where α is an infinite cardinal, if every subspace of X of cardinality $\leq \alpha$ has compact closure in X.) That α-boundedness is closed hereditary and preserved under finite closed sums follows easily from its definition.

Definition 8.5. Let $\mathfrak{P}$ be a topological property. A space X is called *locally-$\mathfrak{P}$* if each $x \in X$ has a neighborhood in X with $\mathfrak{P}$.

Lemma 8.6. *Let X be a regular space and let $\mathfrak{P}$ be a closed hereditary topological property preserved under finite closed sums. Then X is locally null (with respect to the ideal $\mathscr{I}_{\mathfrak{P}}$) if and only if X is locally-$\mathfrak{P}$.*

Proof. Note that $\mathscr{I}_{\mathfrak{P}}$ is an ideal in $(\mathscr{P}(X), \subseteq)$ by Lemma 8.3. Let X be locally null and let $x \in X$. Then $x \in \mathrm{int}_X U$ for some $U \in \mathscr{I}_{\mathfrak{P}}$. Thus $\mathrm{cl}_X U$ is a neighborhood of x in X with $\mathfrak{P}$. For the converse, let $x \in X$ have a neighborhood V in X with $\mathfrak{P}$. There exists an open neighborhood W of x in X with $\mathrm{cl}_X W \subseteq V$. Now, $\mathrm{cl}_X W$ has $\mathfrak{P}$, as it is closed in V and V has $\mathfrak{P}$. Thus W is null. $\square$

We introduce the following notation for convenience.

Notation 8.7. Let X be a space and let $\mathfrak{P}$ be a closed hereditary topological property preserved under finite closed sums. Denote

$$C_{00}^{\mathfrak{P}}(X) = C_{00}^{\mathscr{I}_{\mathfrak{P}}}(X) \quad \text{and} \quad C_{0}^{\mathfrak{P}}(X) = C_{0}^{\mathscr{I}_{\mathfrak{P}}}(X).$$

Also, denote

$$\lambda_{\mathfrak{P}} X = \lambda_{\mathscr{I}_{\mathfrak{P}}} X.$$

Theorem 8.8. *Let X be a space and let $\mathfrak{P}$ be a closed hereditary topological property preserved under finite closed sums. Then*

$$C_{00}^{\mathfrak{P}}(X) = \big\{ f \in C_b(X) : \mathrm{supp}(f) \text{ has a closed neighborhood in } X \text{ with } \mathfrak{P} \big\}.$$

Furthermore,

- *$C_{00}^{\mathfrak{P}}(X)$ is an ideal (in particular, a normed subalgebra) in the algebra $C_b(X)$.*

If X is completely regular, then

- *$C_{00}^{\mathfrak{P}}(X)$ is of empty hull if and only if X is locally-$\mathfrak{P}$.*

If X is completely regular and locally-$\mathfrak{P}$, then

- *$C_{00}^{\mathfrak{P}}(X)$ is unital if and only if X is non-$\mathfrak{P}$.*

If X is normal and locally-$\mathfrak{P}$, then

- *The normed algebra $C^{\mathfrak{P}}_{00}(X)$ is isometrically isomorphic to $C_{00}(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y, namely $Y = \lambda_{\mathfrak{P}}X$. Furthermore,*
  - *X is dense in Y.*
  - *$C^{\mathfrak{P}}_{00}(X)$ is unital if and only if Y is compact.*

Proof. Let $f \in C_b(X)$. If $\mathrm{supp}(f)$ has a closed neighborhood U in X with $\mathfrak{P}$ then $\mathrm{supp}(f)$ has a null neighborhood in X, namely, U itself. Thus $f \in C^{\mathfrak{P}}_{00}(X)$. On the other hand, if $f \in C^{\mathfrak{P}}_{00}(X)$, then $\mathrm{supp}(f)$ has a null neighborhood V in X. Thus $\mathrm{cl}_X V$ is a closed neighborhood of $\mathrm{supp}(f)$ in X with $\mathfrak{P}$. The remaining assertions of the theorem follow from Lemma 8.6 (and Lemma 8.3) and Theorems 3.3, 3.8, 3.10 and 3.13. □

Theorem 8.9. *Let X be a space and let $\mathfrak{P}$ be a closed hereditary topological property preserved under finite closed sums. Then*

$$C^{\mathfrak{P}}_{0}(X) = \big\{f \in C_b(X) : |f|^{-1}\big([1/n, \infty)\big) \text{ has } \mathfrak{P} \text{ for each } n\big\}.$$

Furthermore,

- *$C^{\mathfrak{P}}_{0}(X)$ is a closed ideal (in particular, a Banach subalgebra) in the normed algebra $C_b(X)$.*
- *$C^{\mathfrak{P}}_{00}(X) \subseteq C^{\mathfrak{P}}_{0}(X)$.*

If X is completely regular, then

- *$C^{\mathfrak{P}}_{0}(X)$ is of empty hull if and only if X is locally-$\mathfrak{P}$.*

If X is completely regular and locally-$\mathfrak{P}$, then

- *$C^{\mathfrak{P}}_{0}(X)$ is unital if and only if X is non-$\mathfrak{P}$.*

If X is normal and locally-$\mathfrak{P}$, then

- *The Banach algebra $C^{\mathfrak{P}}_{0}(X)$ is isometrically isomorphic to $C_0(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y, namely $Y = \lambda_{\mathfrak{P}}X$. Furthermore,*
  - *X is dense in Y.*
  - *$C^{\mathfrak{P}}_{00}(X)$ is dense in $C^{\mathfrak{P}}_{0}(X)$.*
  - *$C^{\mathfrak{P}}_{0}(X)$ is unital if and only if Y is compact.*

Proof. If n is a positive integer, then $|f|^{-1}([1/n, \infty))$ (since it is closed in X) is null if and only if it has $\mathfrak{P}$. The given representation of $C^{\mathfrak{P}}_{0}(X)$ follows then from Proposition 4.3 (and Lemma 8.3). The remaining assertions of the theorem follow from Lemma 8.6 and Theorems 4.5, 4.6, 4.7 and 4.9. □

The following introduces conditions under which the representation of $C^{\mathfrak{P}}_{00}(X)$ simplifies.

Theorem 8.10. *Let $\mathfrak{P}$ and $\mathfrak{Q}$ be topological properties such that*

- *$\mathfrak{P}$ is closed hereditary and preserved under countable closed sums.*
- *$\mathfrak{Q}$ is closed hereditary.*
- *Any space with $\mathfrak{Q}$ having a dense subspace with $\mathfrak{P}$ has $\mathfrak{P}$.*

Let X be a space with $\mathfrak{Q}$. Then

(1) *$\mathscr{I}_{\mathfrak{P}}$ is a σ-ideal in $(\mathscr{P}(X), \subseteq)$.*

(2) *If X is regular and locally-$\mathfrak{P}$ and*
$$\mathfrak{P}+\mathfrak{Q}\rightarrow \textit{The Lindelöf property}$$
then
$$C^{\mathfrak{P}}_{00}(X)=\big\{f\in C_b(X):\mathrm{supp}(f)\ \textit{has}\ \mathfrak{P}\big\}.$$
Also,
$$C^{\mathfrak{P}}_{00}(X)=C^{\mathfrak{P}+\mathfrak{Q}}_{00}(X)\quad \textit{and}\quad C^{\mathfrak{P}}_{0}(X)=C^{\mathfrak{P}+\mathfrak{Q}}_{0}(X).$$

Proof. (1). By Lemma 8.3 it suffices to show that $\mathscr{I}_{\mathfrak{P}}$ is closed under countable unions. Let $A_1,A_2,\ldots\in\mathscr{I}_{\mathfrak{P}}$. Then
$$G=\bigcup_{n=1}^{\infty}\mathrm{cl}_X A_n$$
has $\mathfrak{P}$, as each $\mathrm{cl}_X A_1,\mathrm{cl}_X A_2,\ldots$ has $\mathfrak{P}$ (and $\mathfrak{P}$ is preserved under countable close sums). Note that
$$H=\mathrm{cl}_X\bigg(\bigcup_{n=1}^{\infty}A_n\bigg)$$
has $\mathfrak{Q}$, as it is closed in X and X has $\mathfrak{Q}$. Since H contains G as a dense subspace it follows that H has $\mathfrak{P}$. Therefore
$$\bigcup_{n=1}^{\infty}A_n\in\mathscr{I}_{\mathfrak{P}}.$$

(2). Let $f\in C^{\mathfrak{P}}_{00}(X)$. Then $\mathrm{supp}(f)$ has a null neighborhood in X, that is, $\mathrm{supp}(f)\subseteq\mathrm{int}_X U$ for some subspace U of X such that $\mathrm{cl}_X U$ has $\mathfrak{P}$. It then follows that $\mathrm{supp}(f)$ has $\mathfrak{P}$, as it is closed in $\mathrm{cl}_X U$.

Next, let $f\in C_b(X)$ such that $\mathrm{supp}(f)$ has $\mathfrak{P}$. Note that X is locally null by Lemma 8.6. For each $x\in X$ let U_x be a null neighborhood of x in X. Then
$$\mathrm{supp}(f)\subseteq\bigcup_{x\in X}\mathrm{int}_X U_x.$$
Note that $\mathrm{supp}(f)$ has $\mathfrak{Q}$, as it closed in X and X has $\mathfrak{Q}$. Since $\mathrm{supp}(f)$ also has $\mathfrak{P}$, it is then Lindelöf. Let $x_1,x_2,\ldots\in X$ such that
$$\mathrm{supp}(f)\subseteq\bigcup_{n=1}^{\infty}\mathrm{int}_X U_{x_n}.$$
Note that $\mathscr{I}_{\mathfrak{P}}$ is a σ-ideal by (1). Therefore $\bigvee_{n\geq 1}U_{x_n}$ ($=\bigcup_{n\geq 1}U_{x_n}$, by the proof of (1)) is a null neighborhood of $\mathrm{supp}(f)$ in X. Thus $f\in C^{\mathfrak{P}}_{00}(X)$.

To conclude the proof it suffices to show that $\mathscr{I}_{\mathfrak{P}}=\mathscr{I}_{\mathfrak{P}+\mathfrak{Q}}$. It is obvious that $\mathscr{I}_{\mathfrak{P}+\mathfrak{Q}}\subseteq\mathscr{I}_{\mathfrak{P}}$. Let $A\in\mathscr{I}_{\mathfrak{P}}$. Then $\mathrm{cl}_X A$ has $\mathfrak{P}$. But $\mathrm{cl}_X A$ also has $\mathfrak{Q}$, as it is closed in X and X has $\mathfrak{Q}$. Thus $A\in\mathscr{I}_{\mathfrak{P}+\mathfrak{Q}}$. □

In the following we give examples of topological properties $\mathfrak{P}$ and $\mathfrak{Q}$ satisfying the assumption of Theorem 8.10.

Example 8.11. For any open covers $\mathscr{U}$ and $\mathscr{V}$ of a space X we say that $\mathscr{U}$ is a *refinement* of $\mathscr{V}$ if each element of $\mathscr{U}$ is contained in an element of $\mathscr{V}$. An open cover $\mathscr{U}$ of X is called *locally finite* if each point of X has an open neighborhood in X intersecting only a finite number of the elements of $\mathscr{U}$. A regular space X is called *paracompact* if for every open cover $\mathscr{U}$ of X there is an open cover of X

which refines $\mathscr{U}$. Paracompact spaces are generally considered as the simultaneous generalizations of compact Hausdorff spaces and metrizable spaces. Every metrizable space as well as every compact Hausdorff space is paracompact and every paracompact space is normal. (See Theorems 5.1.1, 5.1.3 and 5.1.5 of [7].)

Let $\mathfrak{P}$ be the Lindelöf property and let $\mathfrak{Q}$ be either metrizability or paracompactness. Note that the Lindelöf property is closed hereditary (see Theorem 3.8.4 of [7]) and is preserved under countable closed sums, as obviously, any space which is a union of a countable number of its Lindelöf subspaces is Lindelöf. Note that any subspace of a metrizable space is metrizable and a closed subspace of a paracompact space is paracompact. (See Corollary 5.1.29 of [7].) Also, any paracompact space with a dense Lindelöf space is Lindelöf (see Theorem 5.1.25 of [7]), since any metrizable space is paracompact, it then follows that any metrizable space with a dense Lindelöf space is Lindelöf. Therefore, the pair $\mathfrak{P}$ and $\mathfrak{Q}$ satisfy the assumption of Theorem 8.10.

Note that in the realm of metrizable spaces the Lindelöf property coincides with separability and second countability; denote the latter two by $\mathfrak{S}$ and $\mathfrak{C}$, respectively, and denote the Lindelöf property by $\mathfrak{L}$. Thus, if X is a metrizable space, then

$$\mathscr{I}_{\mathfrak{L}} = \mathscr{I}_{\mathfrak{S}} = \mathscr{I}_{\mathfrak{C}},$$

which implies that

$$C^{\mathfrak{L}}_{00}(X) = C^{\mathfrak{S}}_{00}(X) = C^{\mathfrak{C}}_{00}(X) \quad \text{and} \quad C^{\mathfrak{L}}_{0}(X) = C^{\mathfrak{S}}_{0}(X) = C^{\mathfrak{C}}_{0}(X).$$

Let X be a locally separable metrizable space. The study of the Banach subalgebra $C_s(X)$ of $C_b(X)$ consisting of those elements of $C_b(X)$ with separable support, constitutes the subject matter of our next result. We need some preliminaries first.

The *density* of a space X, denoted by $d(X)$, is defined by

$$d(X) = \min\big\{|D| : D \text{ is dense in } X\big\} + \aleph_0.$$

In particular, a space X is separable if and only if $d(X) = \aleph_0$. Note that in any metrizable space the notions of separability and second countability coincide; thus any subspace of a separable metrizable space is separable. A theorem of Alexandroff states that any locally separable metrizable space X can be represented as a disjoint union

$$X = \bigcup_{i\in I} X_i,$$

where I is an index set, and X_i is a non-empty separable (and thus Lindelöf) open-closed subspace of X for each $i \in I$. (See Problem 4.4.F of [7].) Note that $d(X) = |I|$, if I is infinite.

Let D be an uncountable discrete space. Denote by D_λ the subspace of βD consisting of elements in the closure in D of countable subsets of D. In [28], the author proves the existence of a continuous (2-valued) function $f : D_\lambda\backslash D \to [0, 1]$ which is not continuously extendible over $\beta D\backslash D$. In particular, this implies that D_λ is not normal. (To see this, suppose the contrary. Note that $D_\lambda\backslash D$ is closed in D_λ, as D is locally compact and thus open in βD. By the Tietze–Urysohn Extension Theorem, f is extendible to a continuous bounded function over D_λ, and therefore over βD_λ; note that $\beta D_\lambda = \beta D$, as $D \subseteq D_\lambda \subseteq \beta D$. But this is not possible.)

A theorem of Tarski states that for any infinite set I, there is a collection $\mathscr{I}$ of cardinality $|I|^{\aleph_0}$ consisting of countable infinite subsets of I, such that the intersection of any two distinct elements of $\mathscr{I}$ is finite. (See [12].) Note that the collection

of all subsets of cardinality at most $\mathfrak{m}$ in a set of cardinality $\mathfrak{n} \geq \mathfrak{m}$ has cardinality at most $\mathfrak{n}^{\mathfrak{m}}$.

Observe that if X is a space and D is a subspace of X, then

$$U \cap \mathrm{cl}_X D = \mathrm{cl}_X(U \cap D)$$

for every open-closed subspace U of X. This simple observation will be used below.

Theorem 8.12. *Let X be a locally separable metrizable space. Let*

$$C_s(X) = \big\{f \in C_b(X) : \mathrm{supp}(f) \text{ is separable}\big\}.$$

Then $C_s(X)$ is a Banach algebra isometrically isomorphic to $C_0(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y. Moreover

1. *Y is countably compact.*
2. *Y is non-normal if X is non-separable.*
3. *$C_0(Y) = C_{00}(Y)$.*
4. *$\dim C_s(X) = d(X)^{\aleph_0}$.*

Proof. Let $\mathfrak{P}$ be the Lindelöf property and let $\mathfrak{Q}$ be metrizability. Then, as we have seen in Example 8.11, the pair $\mathfrak{P}$ and $\mathfrak{Q}$ satisfy the assumption of Theorem 8.10. Consider a representation

$$X = \bigcup_{i \in I} X_i,$$

of X, where X_i's are disjoint non-empty Lindelöf open-closed subspaces of X and I is an index set. To simplify the notation, denote

$$H_J = \bigcup_{i \in J} X_i$$

for any $J \subseteq I$. Observe that each H_J is open-closed in X, thus it has open-closed closure in βX. Also

$$\lambda_{\mathfrak{P}} X = \bigcup \{\mathrm{cl}_{\beta X} H_J : J \subseteq I \text{ is countable}\}. \tag{8.1}$$

To show this, let $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X C$ has a neighborhood U in X such that $\mathrm{cl}_X U$ is Lindelöf. Then $\mathrm{cl}_X C$ itself is Lindelöf, as it is closed in $\mathrm{cl}_X U$, and therefore $\mathrm{cl}_X C \subseteq H_J$ for some countable $J \subseteq I$. Thus $\mathrm{cl}_{\beta X} C \subseteq \mathrm{cl}_{\beta X} H_J$. On the other hand, if $J \subseteq I$ is countable, then H_J is a cozero-set in X, as it is open-closed in X, and it is Lindelöf, as it is a countable union of Lindelöf subspaces. Since $\mathrm{cl}_{\beta X} H_J$ is open in βX we have

$$\mathrm{cl}_{\beta X} H_J = \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} H_J \subseteq \lambda_{\mathfrak{P}} X.$$

Note that (3) implies (1). (3). It suffices to show that every σ-compact subspace of $\lambda_{\mathfrak{P}} X$ is contained in a compact subspace of $\lambda_{\mathfrak{P}} X$. Let $A = \bigcup_{n \geq 1} A_n$, where each A_n is compact, be a σ-compact subspace of $\lambda_{\mathfrak{P}} X$. Using (8.1), by compactness we have

$$A_n \subseteq \mathrm{cl}_{\beta X} H_{J_1} \cup \cdots \cup \mathrm{cl}_{\beta X} H_{J_{k_n}} \tag{8.2}$$

for some countable $J_1, \ldots, J_{k_n} \subseteq I$. Let

$$J = \bigcup_{n=1}^{\infty} (J_{k_1} \cup \cdots \cup J_{k_n}). \tag{8.3}$$

Then J is countable and $A \subseteq \mathrm{cl}_{\beta X} H_J$. That is A is contained in the compact subspace $\mathrm{cl}_{\beta X} H_J$ of $\lambda_{\mathfrak{P}} X$.

(2). Let $x_i \in X_i$ for each $i \in I$. Then $D = \{x_i : i \in I\}$ is a closed discrete subspace of X, and since X is non-separable, it is uncountable. Suppose in the contrary that $\lambda_{\mathfrak{P}} X$ is normal. Then, using (8.1), it follows that

$$\lambda_{\mathfrak{P}} X \cap \mathrm{cl}_{\beta X} D = \bigcup \{\mathrm{cl}_{\beta X} H_J \cap \mathrm{cl}_{\beta X} D : J \subseteq I \text{ is countable}\}$$

is normal, as it is closed in $\lambda_{\mathfrak{P}} X$. Now, let $J \subseteq I$ be countable. Since $\mathrm{cl}_{\beta X} H_J$ is open-closed in βX (using the observation preceding the statement of the theorem) we have

$$\mathrm{cl}_{\beta X} H_J \cap \mathrm{cl}_{\beta X} D = \mathrm{cl}_{\beta X}(\mathrm{cl}_{\beta X} H_J \cap D) = \mathrm{cl}_{\beta X}(H_J \cap D) = \mathrm{cl}_{\beta X}\big(\{x_i : i \in J\}\big).$$

But $\mathrm{cl}_{\beta X} D = \beta D$, as D is closed in X and X is normal. Therefore

$$\mathrm{cl}_{\beta X}\big(\{x_i : i \in J\}\big) = \mathrm{cl}_{\beta X}\big(\{x_i : i \in J\}\big) \cap \mathrm{cl}_{\beta X} D = \mathrm{cl}_{\beta D}\big(\{x_i : i \in J\}\big).$$

Thus

$$\lambda_{\mathfrak{P}} X \cap \mathrm{cl}_{\beta X} D = \bigcup \{\mathrm{cl}_{\beta D} E : E \subseteq D \text{ is countable}\} = D_\lambda,$$

contradicting the fact that D_λ is not normal.

(4). Since X is non-separable, I is infinite and $d(X) = |I|$. Let $\mathscr{I}$ be a collection of cardinality $|I|^{\aleph_0}$ consisting of countable infinite subsets of I, such that the intersection of any two distinct elements of $\mathscr{I}$ is finite. Let $f_J = \chi_{H_J}$ for any $J \in \mathscr{I}$. No element in

$$\mathscr{F} = \{f_J : J \in \mathscr{I}\}$$

is a linear combination of other elements (as each element of $\mathscr{I}$ is infinite and each pair of distinct elements of $\mathscr{I}$ has finite intersection). Observe that $\mathscr{F}$ is of cardinality $|\mathscr{I}|$. Thus

$$\dim C_s(X) \geq |\mathscr{I}| = |I|^{\aleph_0} = d(X)^{\aleph_0}.$$

On the other hand, if $f \in C_s(X)$, then $\mathrm{supp}(f)$ is Lindelöf (as it is separable) and thus $\mathrm{supp}(f) \subseteq H_J$, where $J \subseteq I$ is countable; therefore, we may assume that $f \in C_b(H_J)$. Conversely, if $J \subseteq I$ is countable, then each element of $C_b(H_J)$ can be extended trivially to an element of $C_s(X)$ (by defining it to be identically 0 elsewhere). Thus $C_s(X)$ may be viewed as the union of all $C_b(H_J)$, where J runs over all countable subsets of I. Note that if $J \subseteq I$ is countable, then H_J is separable; thus any element of $C_b(H_J)$ is determined by its value on a countable set. This implies that for each countable $J \subseteq I$, the set $C_b(H_J)$ is of cardinality at most $\mathfrak{c}^{\aleph_0} = 2^{\aleph_0}$. There are at most $|I|^{\aleph_0}$ countable $J \subseteq I$. Therefore

$$\begin{aligned} \dim C_s(X) \leq \big|C_s(X)\big| &\leq \Big|\bigcup \big\{C_b(H_J) : J \subseteq I \text{ is countable}\big\}\Big| \\ &\leq 2^{\aleph_0} \cdot |I|^{\aleph_0} = |I|^{\aleph_0} = d(X)^{\aleph_0}. \end{aligned}$$

□

The *Lindelöf number* of a space X, denoted by $\ell(X)$, is defined by

$$\ell(X) = \min\{\mathfrak{n} : \text{any open cover of } X \text{ has a subcover of cardinality} \leq \mathfrak{n}\} + \aleph_0.$$

In particular, a space X is Lindelöf if and only if $\ell(X) = \aleph_0$. Any locally compact paracompact space X may be represented as

$$X = \bigcup_{i \in I} X_i,$$

where X_i's are disjoint non-empty Lindelöf open-closed subspaces of X and I is an index set. (See Theorem 5.1.27 of [7].) Note that $\ell(X) = |I|$ if I is infinite, and $\ell(X) = d(X)$ if X is a locally separable metrizable space.

The next theorem is a corollary of Theorem 8.10 and is dual to Theorem 8.12.

Theorem 8.13. *Let X be a locally Lindelöf paracompact space. Let*

$$C_l(X) = \big\{f \in C_b(X) : \mathrm{supp}(f) \text{ is Lindelöf}\big\}.$$

Then $C_l(X)$ is a Banach algebra isometrically isomorphic to $C_0(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y. Moreover, if X is also locally compact then

1. *Y is countably compact.*
2. *Y is non-normal if X is non-Lindelöf.*
3. *$C_0(Y) = C_{00}(Y)$.*
4. *$\dim C_l(X) = \ell(X)^{\aleph_0}$.*

Proof. Let $\mathfrak{P}$ be the Lindelöf property and let $\mathfrak{Q}$ be paracompactness. By Example 8.11, the pair $\mathfrak{P}$ and $\mathfrak{Q}$ satisfy the assumption of Theorem 8.10. Note that if X is also locally compact then it assumes a representation as given in the proof of Theorem 8.12. The theorem now follows by an argument analogous to the one given in the proof of Theorem 8.12. □

The topological properties considered so far have all been closed hereditary; we now consider pseudocompactness. (Recall that a completely regular space X is called *pseudocompact* if every continuous $f : X \to \mathbb{R}$ is bounded.) Pseudocompactness is not a closed hereditary topological property, however, it is hereditary with respect to regular closed subspaces, that is, every regular closed subspace of a pseudocompact space is pseudocompact. (See Exercise 3.10.F(e) of [7].) What makes pseudocompactness so distinct is the known structure of $\lambda_{\mathscr{U}} X$ (with the precise definition of the ideal $\mathscr{U}$ as given in Notation 8.14).

Recall that $\mathscr{RC}(X)$ denotes the set of all regular closed subspaces of a space X. As noted previously, $(\mathscr{RC}(X), \subseteq)$ is an upper semi-lattice (indeed, a lattice) with $A \vee B = A \cup B$ for any $A, B \in \mathscr{RC}(X)$.

Notation 8.14. Let X be a completely regular space. Denote

$$\mathscr{U} = \big\{A \in \mathscr{RC}(X) : A \text{ is pseudocompact}\big\}.$$

Lemma 8.15. *Let X be a completely regular space. Then $\mathscr{U}$ is an ideal in $(\mathscr{RC}(X), \subseteq)$.*

Proof. Note that $\mathscr{U}$ is non-empty, as it contains $\emptyset$. Let $B \subseteq A$ with $A \in \mathscr{U}$ and $B \in \mathscr{RC}(X)$. Note that B is regular closed in A. (Since B is regular closed in X we have $B = \mathrm{cl}_X U$ for some open subspace U of X. But then U is open in A and $B = \mathrm{cl}_A U$.) Since A is pseudocompact and pseudocompactness is hereditary with respect to regular closed subspaces then B is pseudocompact. That is $B \in \mathscr{U}$. Next, let $C, D \in \mathscr{U}$. Then $C \vee D = C \cup D$ is pseudocompact, as each C and D is so. Thus $C \vee D \in \mathscr{U}$. □

The following is dual to Lemma 8.6.

Lemma 8.16. *Let X be a completely regular space. Then X is locally null (with respect to the ideal $\mathscr{U}$) if and only if X is locally pseudocompact.*

Proof. The proof is similar to that of Lemma 8.6. Observe that (since X is regular) for every $x \in X$ each neighborhood of x in X contains a regular closed neighborhood of x in X, that is, a neighborhood in X of the form $\mathrm{cl}_X U$ where U is open in X. □

Considering the dualities between Lemmas 8.3 and 8.15 and between Lemmas 8.6 and 8.16, one can state and prove results dual to Theorems 3.3, 3.8, 3.10, 3.13, 4.5, 4.6 and 4.7. We will now proceed with determining $\lambda_{\mathscr{U}} X$. We need some preliminaries first.

A completely regular space X is said to be *realcompact* if it is homeomorphic to a closed subspaces of some product $\mathbb{R}^{\mathfrak{m}}$ of the real line. Realcompactness is a closed hereditary topological property. Every regular Lindelöf space (in particular, every compact Hausdorff space) is realcompact. Also, every realcompact pseudocompact space is compact. To every completely regular space X there corresponds a realcompact space υX (called the *Hewitt realcompactification of* X) which contains X as a dense subspace and is such that every continuous $f : X \to \mathbb{R}$ is continuously extendible over υX; we may assume that $\upsilon X \subseteq \beta X$. Note that a completely regular space X is realcompact if and only if $X = \upsilon X$. (See Section 3.11 of [7] for further information.)

The following lemma, which may be considered as a dual result of Lemma 3.9, is due to A.W. Hager and D.G. Johnson in [10]; a direct proof may be found in [6]. (See also Theorem 11.24 of [28].)

Lemma 8.17 (Hager–Johnson [10])**.** *Let X be a completely regular space and let U be open in X. If $\mathrm{cl}_{\upsilon X} U$ is compact then $\mathrm{cl}_X U$ is pseudocompact.*

Observe that any completely regular space X with a dense pseudocompct subspace A is pseudocompact; as for any continuous $f : X \to \mathbb{R}$ we have

$$f(X) = f(\mathrm{cl}_X A) \subseteq \mathrm{cl}_{\mathbb{R}} f(A)$$

and $f(A)$ is bounded (since A is pseudocompact).

Lemma 8.18. *Let X be a completely regular space and let A be regular closed in X. Then $\mathrm{cl}_{\beta X} A \subseteq \upsilon X$ if and only if A is pseudocompact.*

Proof. One half follows from Lemma 8.17, as if $\mathrm{cl}_{\beta X} A \subseteq \upsilon X$ then $\mathrm{cl}_{\upsilon X} A = \mathrm{cl}_{\beta X} A$ is compact, as it is closed in βX. For the other half, note that if A is pseudocompact then so is $\mathrm{cl}_{\upsilon X} A$, as it contains A as a dense subspace. But $\mathrm{cl}_{\upsilon X} A$ is realcompact (as it is closed in υX and υX is so) and therefore is compact. Thus $\mathrm{cl}_{\beta X} A \subseteq \mathrm{cl}_{\upsilon X} A$. □

Theorem 8.19. *Let X be a completely regular space. Then*

$$\lambda_{\mathscr{U}} X = \mathrm{int}_{\beta X} \upsilon X.$$

Proof. Suppose that $C \in \mathrm{Coz}(X)$ is such that $\mathrm{cl}_X C$ has a pseudocompact neighborhood U in X. Since U is pseudocompact and $\mathrm{cl}_X C$ is regular closed in X (and thus in U) then $\mathrm{cl}_X C$ is pseudocompact. Thus $\mathrm{cl}_{\beta X} C \subseteq \upsilon X$, by Lemma 8.18, and therefore $\mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} C \subseteq \mathrm{int}_{\beta X} \upsilon X$.

To show the reverse inclusion, let $t \in \mathrm{int}_{\beta X} \upsilon X$. By the Urysohn Lemma (and since βX is normal, as it is compact Hausdorff) there is a continuous $f : \beta X \to [0, 1]$ with $f(t) = 0$ and $f|_{\beta X \setminus \mathrm{int}_{\beta X} \upsilon X} \equiv \mathbf{1}$. Then

$$C = X \cap f^{-1}\big([0, 1/2)\big) \in \mathrm{Coz}(X)$$

and $t \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C$, as $t \in f^{-1}([0,1/2))$ and $f^{-1}([0,1/2)) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C$ by Lemma 3.7. Note that if $V = X \cap f^{-1}([0,2/3))$ then $\mathrm{cl}_X V$ is a neighborhood of $\mathrm{cl}_X C$ in X, and $\mathrm{cl}_X V$ is pseudocompact by Lemma 8.18, as it is regular closed in X and

$$\mathrm{cl}_{\beta X}V \subseteq f^{-1}\big([0,2/3]\big) \subseteq \upsilon X.$$

□

In our final result we will be dealing with realcompactness. Despite the fact that realcompactness is closed hereditary, it is unfortunately not preserved under finite closed sums in the realm of completely regular spaces. (In [21] – a correction in [22] – S. Mrówka describes a completely regular space which is not realcompact but it can be represented as the union of two of its closed realcompact subspaces; a simpler example is given by A. Mysior in [23].) So, our previous results are not applicable as long as the underlying space is required to be only completely regular. As we will see, the situation changes if we confine ourselves to the class of normal spaces.

Recall that a subspace A of a space X is called *C-embedded in X* if every continuous $f : A \to \mathbb{R}$ is continuously extendible over X.

Lemma 8.20 (Gillman–Jerison [9])**.** *Let X be a completely regular space. If A is C-embedded in X then* $\mathrm{cl}_{\upsilon X}A = \upsilon A$.

By $\mathscr{Q}$ in the following we simply mean $\mathscr{I}_{\mathfrak{P}}$, as defined in Notation 8.2, with $\mathfrak{P}$ assumed to be realcompactness; the re-definition is for convenience. (This is also consistent with the initial terminology once used to refer to realcompact spaces. Realcompact spaces were originally introduced by E. Hewitt in [11]; they were then called *Q-spaces*.)

Notation 8.21. Let X be a space. Denote

$$\mathscr{Q} = \{A \subseteq X : \mathrm{cl}_X A \text{ is realcompact}\}.$$

Recall that a completely regular space X is realcompact if and only if $X = \upsilon X$. Observe that in a normal space each closed subspace is C-embedded. (See Problem 3.D.1 of [9].) This observation will be used in the following.

Lemma 8.22. *Let X be a normal space. Then $\mathscr{Q}$ is an ideal in $(\mathscr{P}(X), \subseteq)$.*

Proof. Note that $\mathscr{Q}$ is non-empty, as it contains $\emptyset$. Let $B \subseteq A$ with $A \in \mathscr{Q}$. Then $\mathrm{cl}_X B \subseteq \mathrm{cl}_X A$. Since $\mathrm{cl}_X A$ is realcompact and realcompactness is closed hereditary then $\mathrm{cl}_X B$ is realcompact. That is $B \in \mathscr{Q}$. Next, let $C, D \in \mathscr{Q}$. Since X is normal, every closed subspace of X is C-embedded in X. Thus, using Lemma 8.20 we have

$$\begin{aligned} \upsilon\big(\mathrm{cl}_X(C \cup D)\big) &= \mathrm{cl}_{\upsilon X}(C \cup D) \\ &= \mathrm{cl}_{\upsilon X}C \cup \mathrm{cl}_{\upsilon X}D \\ &= \upsilon(\mathrm{cl}_X C) \cup \upsilon(\mathrm{cl}_X D) \\ &= \mathrm{cl}_X C \cup \mathrm{cl}_X D = \mathrm{cl}_X(C \cup D). \end{aligned}$$

That is $\mathrm{cl}_X(C \cup D)$ is realcompact. Therefore $C \cup D \in \mathscr{Q}$. □

Once one states and proves a lemma dual to Lemma 8.6 it will be then possible to state and prove results dual to Theorems 3.3, 3.8, 3.10, 3.13, 4.5, 4.6 and 4.7. Our concluding result determines $\lambda_{\mathscr{Q}}X$. As in the case of pseudocompactness, it turns out that $\lambda_{\mathscr{Q}}X$ is a familiar subspace of βX as well.

Theorem 8.23. *Let X be a normal space. Then*
$$\lambda_{\mathscr{Q}} X = \beta X \backslash \mathrm{cl}_{\beta X}(\upsilon X \backslash X).$$

Proof. Suppose that $C \in \mathrm{Coz}(X)$ is such that $\mathrm{cl}_X C$ has a realcompact neighborhood U in X. Then $\mathrm{cl}_X C$ is realcompact, as it is closed in U. Since $\mathrm{cl}_X C$ is C-embedded in X, as X is normal, by Lemma 8.20 we have $\mathrm{cl}_{\upsilon X} C = \upsilon(\mathrm{cl}_X C) = \mathrm{cl}_X C$. But then $\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} C \cap (\upsilon X \backslash X)$ is empty, as
$$\mathrm{cl}_{\beta X} C \cap (\upsilon X \backslash X) = \mathrm{cl}_{\upsilon X} C \cap (\upsilon X \backslash X) = \emptyset$$
and thus $\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} C \cap \mathrm{cl}_{\beta X}(\upsilon X \backslash X)$ is empty, that is
$$\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} C \subseteq \beta X \backslash \mathrm{cl}_{\beta X}(\upsilon X \backslash X).$$

To show the reverse inclusion, let $t \in \beta X \backslash \mathrm{cl}_{\beta X}(\upsilon X \backslash X)$. Let $f : \beta X \to [0,1]$ be continuous with $f(t) = 0$ and $f|_{\mathrm{cl}_{\beta X}(\upsilon X \backslash X)} \equiv \mathbf{1}$. Then
$$C = X \cap f^{-1}\big([0,1/2)\big) \in \mathrm{Coz}(X)$$
and $t \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} C$, as $t \in f^{-1}([0,1/2))$ and $f^{-1}([0,1/2)) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} C$ by Lemma 3.7. Now let $V = X \cap f^{-1}([0,2/3))$. Then $\mathrm{cl}_X V$ is a neighborhood of $\mathrm{cl}_X C$ in X. Since $\mathrm{cl}_{\beta X} V \cap (\upsilon X \backslash X)$ is empty, as $\mathrm{cl}_{\beta X} V \subseteq f^{-1}([0,2/3])$, we have
$$\mathrm{cl}_X V = X \cap \mathrm{cl}_{\beta X} V = \upsilon X \cap \mathrm{cl}_{\beta X} V = \mathrm{cl}_{\upsilon X} V.$$
Therefore $\mathrm{cl}_X V$ is realcompact, as it is closed in υX. $\square$

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